Session Objectives

- Define coordinate systems and vectors
- Add and subtract vectors; compute the dot product and vector length
- Define a unit vector
- Compute the angle between two vectors
- Implement vector functions in Ada

Geometry

Ptolemy I, a pharaoh of Egypt, decided to take some lessons in mathematics from Euclid. When he complained that a certain proposition was too difficult for him to understand, Euclid replied:

> There is no royal road to geometry.

What he meant was that mathematics is how it is, and just because someone is in a privileged position, they do not get a short cut to the truth.

Why Vectors?

- Many variables can be represented by a simple number, or scalar: for instance, time, distance, speed.
- However, many quantities are better represented by their size (or magnitude) and a direction. For example, velocity (of missiles?), acceleration, force, etc.
- Most computer models of the real world (for example, games, simulations, virtual reality) require us to deal with a multi-dimensional space, as objects are rarely points moving conveniently along a single line.

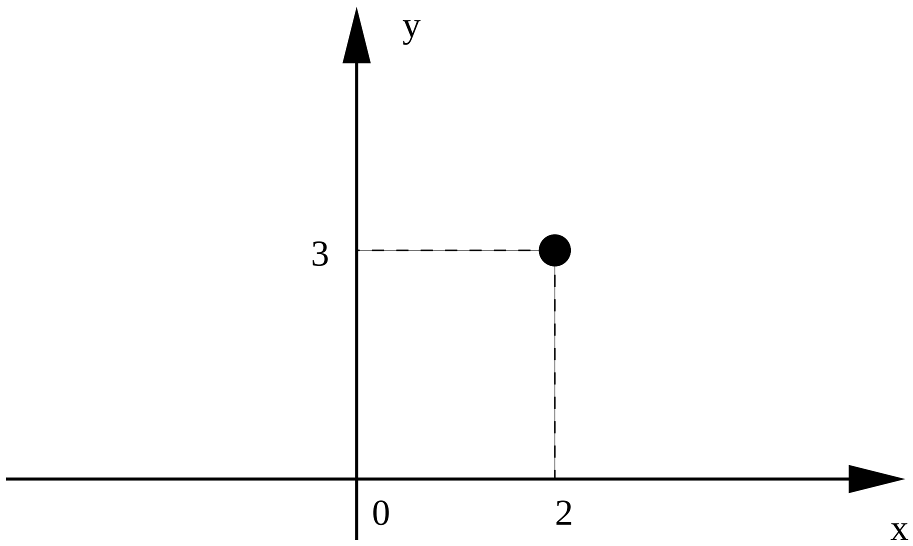
Coordinate Systems

- A vector is an n-tuple of numbers representing a point in an n-dimensional space:

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Vectors are usually written as columns, though sometimes rows are more convenient to save space.

- A vector in one-dimensional space is a simply a number, and is called a scalar.

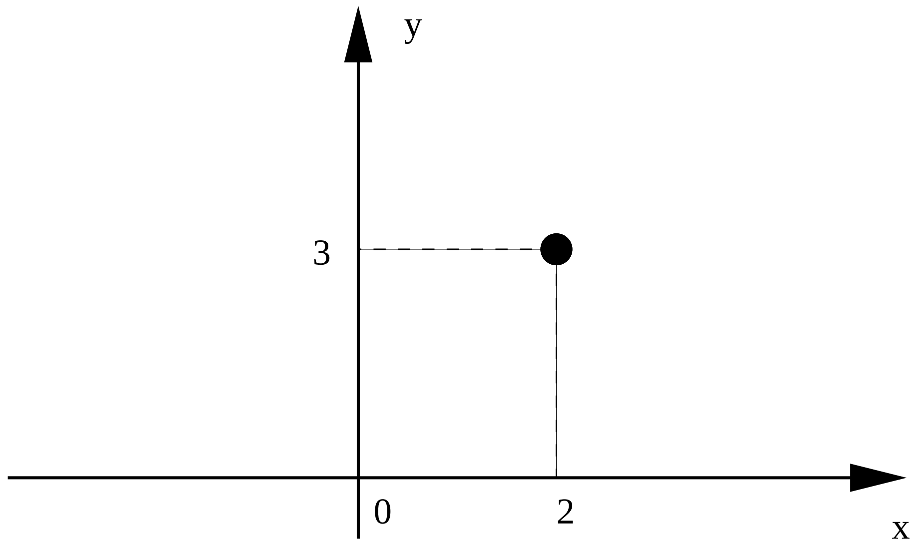


The vector $\mathbf{v}$ represents a point in a two-dimensional space.

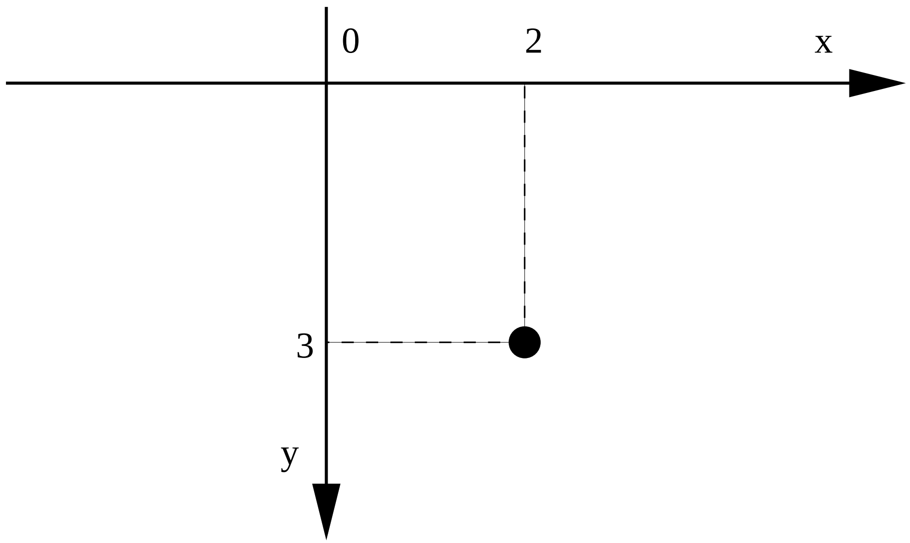
Vector Components

- We choose orthogonal straight lines through the origin O as our coordinate system (so the axes are at right angles).
- Each component of a vector is given by the corresponding distance along the axis of a perpendicular dropped to that axis.
- The origin is represented by the zero vector, which is a vector of length n all of whose components are zero.
- In mathematics, the x-axis is usually drawn from left to right and the y-axis from bottom to top. However, in computer graphics, the origin is often chosen at the top left-hand corner of the screen with the x-axis from left to right and the y-axis top to bottom.

Axes Conventions



Mathematical Axes



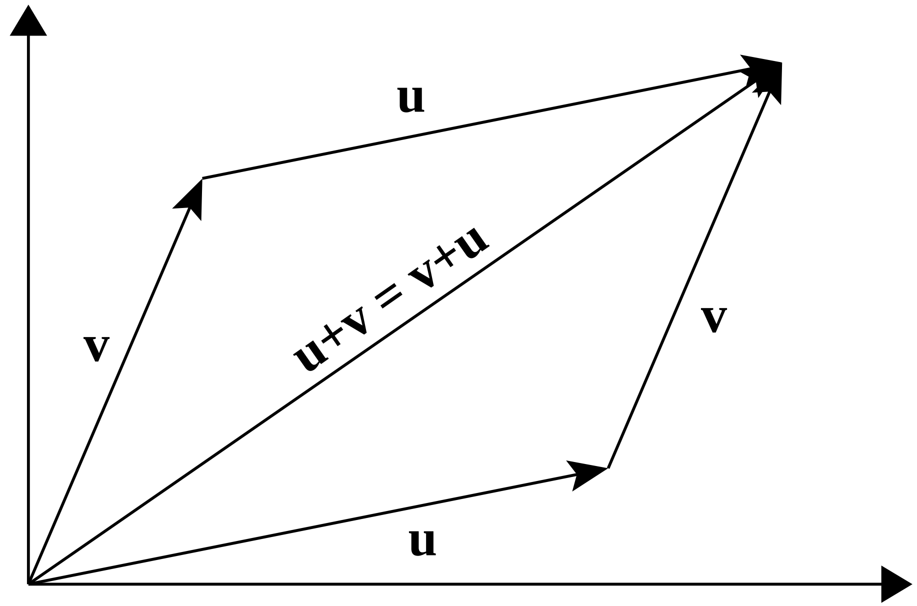
Computer Graphics Axes

Vector Addition

To add two vectors, which must have the same dimensionality, add the corresponding elements of the vectors.

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

For example, $[1,2,3]+[-1,0,2]=[0,2,5]$.



Vector Arithmetic

Subtraction To subtract one vector from another, subtract the corresponding elements of the vectors. Both vectors must have the same dimensionality. The vector $\mathbf{v} - \mathbf{u}$ represents the translation vector that takes the point $\mathbf{u}$ to the point $\mathbf{v}$. For example, to get from $[1, 2]$ to $[0, 3]$ we translate by $[0, 3] - [1, 2] = [-1, 1]$.

Scalar Multiplication To multiply a vector by a scalar, multiply every element of the vector by the scalar.

$$s\mathbf{v} = s \times \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} sv_1 \\ sv_2 \\ \vdots \\ sv_n \end{bmatrix} \tag{1}$$

For example, $2 \times [1, 2, 3] = [2, 4, 6]$. This gives another vector in the same direction as $\mathbf{v}$ but with a different size.

Exercise

If $\mathbf{u} = [2, 3, 3]$, $\mathbf{v} = [-2, -3, 1]$, $\mathbf{w} = [1, 2]$ and $\mathbf{x} = [-2, 0]$ compute all possible vector sums and differences.

Solution

If $\mathbf{u} = [2, 3, 3]$, $\mathbf{v} = [-2, -3, 1]$, $\mathbf{w} = [1, 2]$ and $\mathbf{x} = [-2, 0]$ compute all possible vector sums and differences.

Solution: We can only add or subtract vectors with the same dimensionality. Hence $\mathbf{u}$ and $\mathbf{v}$ can be combined, and $\mathbf{w}$ and $\mathbf{x}$ can be combined.

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

$$\mathbf{v} - \mathbf{u} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ -2 \end{bmatrix}$$

$$\mathbf{w} + \mathbf{x} = \mathbf{x} + \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\mathbf{w} - \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\mathbf{x} - \mathbf{w} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

Exercise

Calculate $-1 \times \mathbf{u}$, $0 \times \mathbf{v}$ and $2 \times \mathbf{x}$.

Solution

Calculate $-1 \times \mathbf{u}$, $0 \times \mathbf{v}$ and $2 \times \mathbf{x}$.

Solution:

$$-1 \times \mathbf{u} = -1 \times \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -3 \end{bmatrix}$$

$$0 \times \mathbf{v} = 0 \times \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2 \times \mathbf{x} = 2 \times \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

Dot Product

- There is no general definition of multiplication or division of two vectors.
- Given two vectors $\mathbf{u}$ and $\mathbf{v}$ of the same dimensionality, we define their dot product (also called their inner product) to be:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i. \qquad (2)$$

  For example, the dot product of $[1, 2, 3]$ and $[-1, 2, 0]$ is $1 \times -1 + 2 \times 2 + 3 \times 0 = 3$.

- It is clear that the dot product is commutative (that is, the order is not important), as $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

Length and Distance

- The length or magnitude of a vector $\mathbf{v}$, written $\|\mathbf{v}\|$, is given by the square root of the dot product of the vector with itself $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sum_{i=1}^{n} v_i^2$.
- For example, the length of $[3,4]$ is $\sqrt{3^2+4^2} = \sqrt{25} = 5$.
- This can be interpreted as the distance from the origin of the point represented by $\mathbf{v}$.
- From the formula we can immediately deduce that $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$, the zero vector.
- The distance between two points $\mathbf{u}$ and $\mathbf{v}$ is the length of the vector joining them, which is $\|\mathbf{v}-\mathbf{u}\|$:

$$\|\mathbf{v}-\mathbf{u}\| = \sqrt{\sum_{i=1}^{n}(v_i-u_i)^2}. \tag{3}$$

Exercise

For vectors $\mathbf{u} = [2, 3, 3]$ and $\mathbf{v} = [-2, -3, 1]$ compute $\mathbf{u} \cdot \mathbf{v}$, the distance between $\mathbf{u}$ and $\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$. (You may leave square roots in their algebraic form (i.e. you don't have to write down the numeric value if you don't have a calculator)).

Solution

For vectors $\mathbf{u} = [2, 3, 3]$ and $\mathbf{v} = [-2, -3, 1]$ compute $\mathbf{u} \cdot \mathbf{v}$, the distance between $\mathbf{u}$ and $\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

Solution:

$$\mathbf{u} \cdot \mathbf{v} = 2 \times -2 + 3 \times -3 + 3 \times 1 = -4 - 9 + 3 = -10$$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{\|[4, 6, 2]\|} = \sqrt{4^2 + 6^2 + 2^2} = \sqrt{56} \approx 7.4833$$

$$\|\mathbf{u}\| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22} \approx 4.6904$$

$$\|\mathbf{v}\| = \sqrt{(-2)^2 + (-3)^2 + 1^2} = \sqrt{14} \approx 3.7417$$

Unit Vectors

A unit vector is a vector of length 1. The unit vector in the direction of $\mathbf{v}$ is given by $\mathbf{v}/\|\mathbf{v}\|$ (provided $\mathbf{v}$ is not the zero vector).

Prove that the length of $\mathbf{v}/\|\mathbf{v}\|$ is equal to 1. (Hint: write $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, $\mathbf{w} = \mathbf{v}/\|\mathbf{v}\|$ and calculate $\|\mathbf{w}\|$).

Solution

Let $\mathbf{w} = \mathbf{v}/\|\mathbf{v}\|$. So the ith coordinate $w_i = v_i/\|\mathbf{v}\|$. Then

$$\|\mathbf{w}\| = \sqrt{\mathbf{w} \cdot \mathbf{w}} = \sqrt{\sum_{i=1}^{n} w_i^2}$$

$$= \sqrt{\sum_{i=1}^{n} (v_i/\|\mathbf{v}\|)^2}$$

$$= \sqrt{\frac{1}{\|\mathbf{v}\|^2} \sum_{i=1}^{n} v_i^2}$$

$$= \sqrt{\frac{\|\mathbf{v}\|^2}{\|\mathbf{v}\|^2}}$$

$$= 1.$$

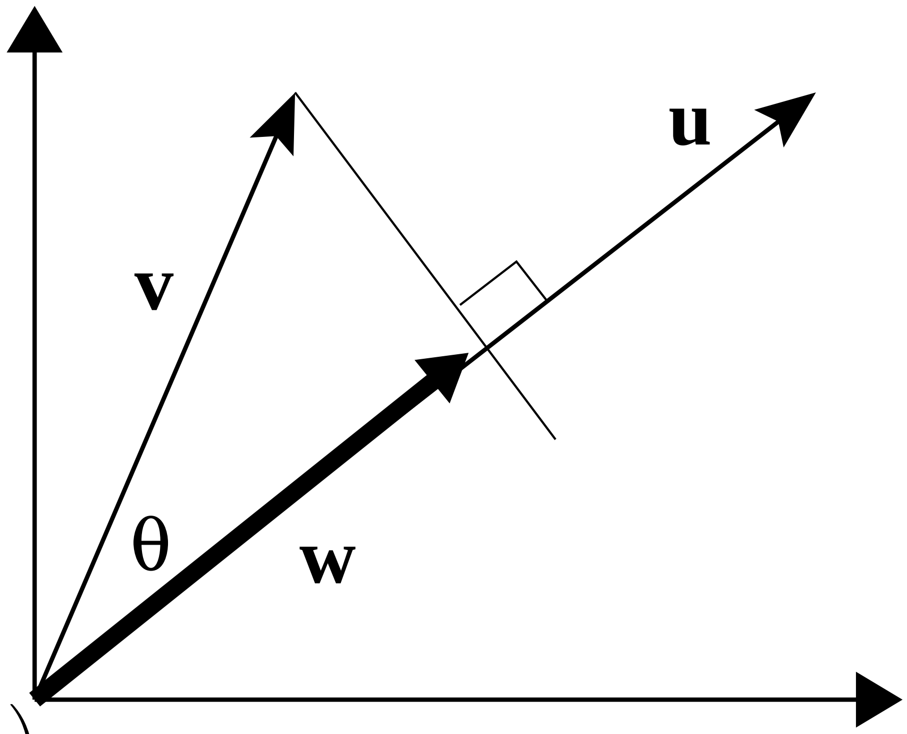
Angles

- To calculate the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\theta = \arccos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

- There is a geometric interpretation of the dot product in terms of projecting one vector $\mathbf{v}$ onto a unit vector $\mathbf{u}$.

- The length of the projection $\mathbf{w}$ is given by



$$\|\mathbf{w}\| = \|\mathbf{v}\|\cos(\theta) = \|\mathbf{v}\|\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|\|\mathbf{u}\|}\right) = \mathbf{v}\cdot\mathbf{u}.$$

Vectors in Ada

- The most appropriate Ada data structure to represent a vector is the unconstrained array.

```
TYPE Vector IS ARRAY(RANGE <>) OF Real;
```

- It is straightforward to write functions that compute the effect of operations on vectors. Dot product:

```
1  FUNCTION DotProduct(U, V: Vector) RETURN Real IS
2    Result : Real := 0.0;
3  BEGIN
4    IF U'First /= V'First OR U'Last /= V'Last THEN
5      RAISE Constraint_Error;
6    END IF;
7    FOR I IN U'Range LOOP
8      Result := Result + U(I)*V(I);
9    END LOOP;
10   RETURN result;
11 END DotProduct;
```

Note how in lines 4–6 we check that the bounds of the two vectors are the same, and raise an exception if they are not.

Exercise

Write an Ada function that computes the sum of two vectors. (Hint: the return type can be defined as `R : Vector(U'Range)`, where `U` is one of the vector parameters).

Solution

Write an Ada function that computes the sum of two vectors.

```
FUNCTION VecSum(U, V: Vector) RETURN Vector IS
  R : Vector(U'Range);
BEGIN
  IF U'First /= V'First OR U'Last /= V'Last THEN
    RAISE Constraint_Error;
  END IF;
  FOR I IN U'Range LOOP
    R(I) := U(I)+V(I);
  END LOOP;
  RETURN R;
END VecSum;
```

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