Session Objectives

- Define polynomials and rational functions
- Use Horner's algorithm to evaluate polynomials efficiently
- Define and use recurrence relations
- Use series to evaluate elementary functions

Polynomials

- A polynomial is a function of a single variable x; it consists of a weighted sum of non-negative powers of x:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i. \tag{1}$$

- The coefficients a_0, ..., a_n are real numbers, and $a_n \neq 0$ (though any other coefficient may be zero). The value a_0 is known as the constant term.
- The degree of the polynomial is n. Linear functions are polynomials of degree 1 and quadratic functions are polynomials of degree 2.
- The growth of the polynomial as $x \to \infty$ is dominated by the largest power of x, that is, $a_n x^n$.

Polynomial Arithmetic

The space of polynomials is closed under addition, subtraction, and multiplication. If $p_1(x) = \sum_{i=0}^{m} a_i x^i$ and $p_2(x) = \sum_{j=0}^{n} b_j x^j$ are both polynomials, we define their sum, difference and product as follows:

$$(p_1 + p_2)(x) = \sum_{i=0}^{\max(m,n)} (a_i + b_i)x^i \tag{2}$$

$$(p_1 - p_2)(x) = \sum_{i=0}^{\max(m,n)} (a_i - b_i)x^i \tag{3}$$

$$(p_1 \times p_2)(x) = \sum_{i=0}^{m+n} c_i x^i \qquad \text{where } c_i = \sum_{k=0}^{i} a_k b_{i-k}. \tag{4}$$

Exercise

For example, if $p_1(x) = 2x - 1$ and $p_2(x) = x^2 - x + 3$, then

$$(p_1 + p_2)(x) = x^2 + x + 2$$

$$(p_1 - p_2)(x) = -x^2 + 3x - 4$$

$$(p_1 \times p_2)(x) = 2x^3 - 3x^2 + 7x - 3.$$

Let $p_1(x) = 3x^2 - 1$ and $p_2(x) = x^2 + 2x$. Compute

1. $p_1 + p_2$
2. $p_1 - p_2$
3. $p_1 \times p_2$

Solution

Let $p_1(x) = 3x^2 - 1$ and $p_2(x) = x^2 + 2x$. Compute

1. $p_1 + p_2 = 4x^2 + 2x - 1$.
2. $p_1 - p_2 = 2x^2 - 2x - 1$.
3. $p_1 \times p_2 = 3x^4 + 6x^3 - x^2 - 2x$.

Horner's Algorithm

- We need to evaluate polynomials.
- Just taking the definition and computing x^2, . . . , x^n by repeated multiplications (or, even worse, using the power operator `**`) is terribly inefficient.
- Horner's algorithm requires just n multiplications and $n+1$ additions. Assume that the array `A` contains the coefficients from `A(0..n)`. The algorithm can be written as

```
p = A'LAST;
FOR I IN REVERSE A'FIRST..(A'LAST-1) LOOP
  p = p * x + A(I);
END LOOP;
```

Summary

1. A polynomial is a function of a single variable x that consists of a weighted sum of non-negative powers of x. Rational functions are the set of ratios between polynomials.
2. The space of polynomials is closed under addition, subtraction, and multiplication.
3. Horner's algorithm is an efficient way of calculating polynomials.

Sequences and Series

- Sequences have two main applications: they are a digital representation of a signal after analog to digital conversion and they are a method for solving numerical problems by getting a sequence of answers, each being closer to the true solution than the last.
- Sequences are often defined in the form of a recurrence relation, which defines an element in terms of a finite number of previous elements. The Fibonacci sequence $a_n = a_{n-1} + a_{n-2}$ is a well-known example.
- The sum of a sequence of terms is called a series.
- An important example of a series is the Taylor series which can be used to approximate a function, and is the basis of algorithms for computing trigonometric, exponential and logarithmic functions.

Sequences

- A sequence is a list of numbers. Some examples of sequences are

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

$$1, -1, 1, -1, 1, -1, 1, \ldots$$

$$2, 3, 5, 7, 11, 13, 17, 19, \ldots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- A sequence may be finite (as in the first example), or infinite (as in the other three examples), which is denoted using dots.
- We will often list the elements of a sequence using letters and indices: a_1, a_2, a_3, ... Sometimes it is more convenient to start from an index of zero: a_0.

Defining a Sequence

- It is a good idea to write down a formula for the ith term in a sequence, since this makes the definition precise.
- The first sequence may be written as $\{a_i\}_{i=1}^{10}$ with $a_i = i$.
- The second sequence may be written as $\{a_i\}_{i=0}^{\infty}$ (where the symbol ∞ 'infinity' means that the list goes on forever) with $a_i = (-1)^i$.
- This is not always possible: the third sequence is the list of prime numbers, and there is no known formula to generate the ith prime. (It is a nice theorem that there are an infinite number of primes).
- Write down a formula for the fourth sequence.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Limits

- We are often interested in the limit of a sequence. The basic principle is that the difference between the nth term and some real number should get smaller as n gets larger.
- The values of the fourth sequence become closer and closer to 0. In fact, they can be made arbitrarily close to 0; and we say that the limit of the sequence is 0 (even though none of the terms in the sequence is actually equal to 0; this is a simple example of the slightly tricky nature of limits). We write the limit as $\lim_{n\to\infty} a_n = 0$.
- On the other hand, the second sequence oscillates back and forth between -1 and $+1$; it never gets any closer to any real number, and hence this sequence does not have a limit.
- A sequence like the primes which gets larger and larger is said to be unbounded.

Recurrence Relations

- Another way of defining a sequence is in terms of a recurrence relation, so that $a_n = f(a_{n-1}, \ldots, a_{n-r})$ for some function f and fixed window (known as the degree or order of the recurrence) r. To define the sequence, the first r terms also have to be determined.

- The Fibonacci sequence is defined by the formula $a_n = a_{n-1} + a_{n-2}$ and the first two terms, usually fixed as $a_1 = 1$ and $a_2 = 1$. Then

$$
\begin{aligned}
a_1 &= 1 & a_4 &= a_3 + a_2 = 2 + 1 = 3 \\
a_2 &= 1 & a_5 &= a_4 + a_3 = 3 + 2 = 5 \\
a_3 &= a_2 + a_1 = 1 + 1 = 2 & a_6 &= a_5 + a_4 = 5 + 3 = 8
\end{aligned}
$$

- Calculate a_7 and a_8 for the Fibonacci sequence.

Factorial Function

The factorial function, written $n!$, is defined for natural numbers as follows:

$$n! = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 2 \times 3 \times \cdots \times n & n > 1 \end{cases}$$

So $3! = 2 \times 3 = 6$ and $4! = 2 \times 3 \times 4 = 24$. The factorial function grows very quickly. It can be viewed as a sequence with the following recurrence relation:

$$a_0 = 1$$

$$a_1 = 1$$

$$a_n = n \times a_{n-1} \qquad \text{for } n > 1$$

Series

- A series is the sum of a sequence of numbers.
- If the series contains a finite number of terms then it is a finite series, otherwise it is an infinite series. For example,

$$1+2+3+\cdots+10=\sum_{n=1}^{10} n$$

is a finite series, while

$$1+\frac{1}{2}+\cdots+\frac{1}{2^n}+\cdots=\sum_{n=0}^{\infty} 2^{-n}$$

is an infinite series.

- To calculate an infinite series, form a sequence of 'partial sums' $S_N=\sum_{n=1}^{N} a_n$ and find their limit. If it exists and is finite, then the series is well-defined. For example, the infinite series given above is equal to 2. A series that evaluates to a well-defined finite value is said to converge.

Power Series

A power series is like an infinite polynomial:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=1}^{\infty} a_n x^n. \tag{5}$$

The converence of this series depends on the coefficients a_n and may depend on the value of x as well.

Power series are interesting for two reasons:

1. It is possible to derive convergent power series for many interesting functions using calculus. This is called the Taylor series for the function.
2. Summing the first few terms in a power series (computing S_N for suitable N) can give a good approximation to the infinite series. This often yields a good algorithm for calculating these functions.

Useful Power Series

Sine

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \qquad (6)$$

This series converges for all values of x.

Cosine

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \qquad (7)$$

This series converges for all values of x.

Exponential

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{8}$$

This series converges for all values of x. Substituting $x = 1$ gives the following identity:

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{9}$$

Logarithm

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \tag{10}$$

This series only converges for $-1 < x \leq 1$. Note that it is defined for the natural logarithm (i.e. to base e). To use this power series for larger values of x, use the fact that $\ln(ax) = \ln(a) + \ln(x)$.

Computing Elementary Functions

- We shall find sin(0.1) correct to five decimal places using its power series expansion, substituting $x = 0.1$ and calculating terms until the next term is small compared to 5×10^{-6}.
- We are assuming that the terms are rapidly getting smaller, so the error in our calculation is approximately equal to the first term that we leave out of the infinite sum.

$$\begin{aligned}\sin(0.1) &= 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \cdots \\ &= 0.1 - 0.0001\dot{6} + 0.0000008\dot{3} - \cdots \\ &= 0.09983 \qquad \text{to five decimal places}\end{aligned}$$

Note that the third term in the sum is smaller than 5×10^{-6}, so only the first two terms are used in the final answer.

Exercise

Calculate $e^{0.1}$ to five decimal places.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Solution

Calculate $e^{0.1}$ to five decimal places.

$$
\begin{aligned}
e^{0.1} &= 1 + 0.1 + \frac{0.01}{2!} + \frac{0.001}{3!} + \frac{0.0001}{4!} + \cdots \\
&= 1 + 0.1 + 0.005 + 0.0001\dot{6} + 0.0000041\dot{6} + \cdots \\
&\approx 1 + 0.1 + 0.005 + 0.0001\dot{6} \\
&= 1.1051\dot{6}
\end{aligned}
$$

Implementation of Power Series

- The small number of terms needed in these examples is also dependent on the size of x. If $0 < x < 1$, then $x^n \to 0$ as $n \to \infty$, and hence the terms become small relatively quickly.
- On the other hand, if x is large, then we might need a large number of terms from the Taylor series for an accurate approximation. We rescale x to a smaller value using the properties of the function.
- For example, sin is periodic, so we can subtract multiples of 2π from x until it is in the range $-\pi < x \leq \pi$. We have already discussed the need for such rescaling for $\ln(1+x)$.
- Issues of this type make the implementations of these algorithms best left to experts, but they are all based on a suitable power series expansion.

Summary

1. A sequence is a list of numbers.
2. We write the limit of a sequence a_n as $\lim_{n\to\infty} a_n$.
3. A recurrence relation of degree r has the form $a_n = f(a_{n-1}, \ldots, a_{n-r})$ for some function f.
4. A series is the sum of a sequence of numbers.
5. To calculate an infinite series, we form a sequence of 'partial sums' $S_N = \sum_{n=1}^{N} a_n$ and look at the limit of that sequence.
6. A power series has the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=1}^{\infty} a_n x^n.$$

7. Summing the first few terms in a power series (computing S_N for suitable N) can give a good approximation to the infinite series.
8. The trigonometric, exponential and logarithmic functions are calculated using power series.

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