Session Objectives

- Define elementary functions
- Compute trigonometric functions in degrees and radians
- Explain the relationship between exponentiation and logarithms
- Define natural logarithms and the number e

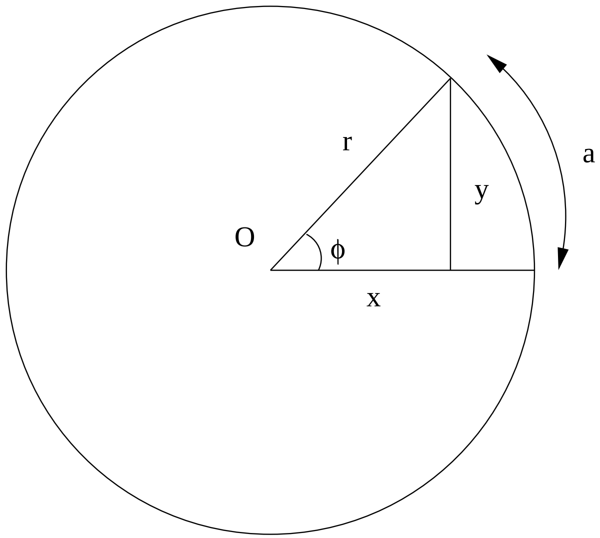
Elementary Functions

- The term elementary functions has a precise technical meaning in mathematics
- it includes any function that involves only the arithmetic operations, trigonometric functions and their inverses, and logarithms and exponentiation.
- They are called ‘elementary’ because they are reasonably easy to deal with and are constructed from a small core of important elements.
- In this section we shall consider in turn the trigonometric functions, logarithms and powers, and polynomials and rational functions.

Trigonometric Functions

- Vital tools for doing geometry.
- If you need to know how long shadows are, how light will look when reflected off objects, how moving objects bounce when they hit the ground, you need trigonometry.
- They are also important analytical tools in the study of waves: not just sea waves (which is a bit specialised) but, using the Fourier transform, sound waves (for musical analysis and synthesis), and physical systems (such as vibration, heat, and electromagnetism).

Definitions



$$\sin(\phi) = \frac{y}{r}$$

$$\cos(\phi) = \frac{x}{r}$$

$$\tan(\phi) = \frac{y}{x} = \frac{\sin(\phi)}{\cos(\phi)}.$$

The angle can now be as large as we like, or negative, and the lengths can also be negative: for example, if $90^\circ < \phi < 180^\circ$, then x is negative, so $\cos(\phi) < 0$ and $\tan(\phi) < 0$, but $\sin(\phi) > 0$.

Trigonometric Identity

We can use Pythagoras' theorem: from the Figure we see that in the right-angled triangle we have $x^2 + y^2 = r^2$. Dividing both sides by r^2, we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

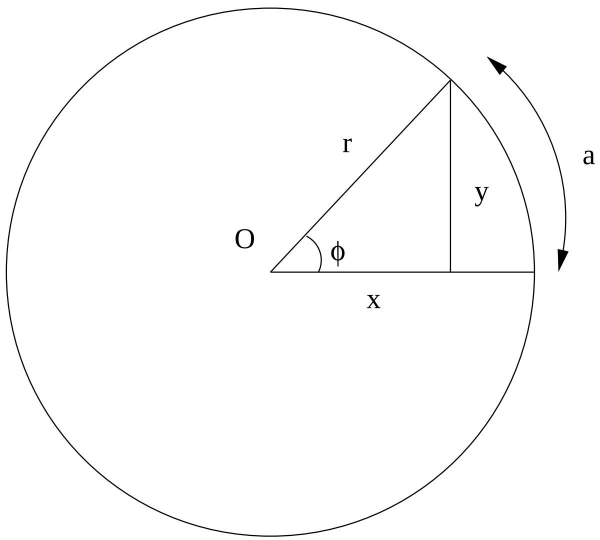
and using the definitions of cos and sin, we get

$$(\cos(\phi))^2 + (\sin(\phi))^2 = 1. \tag{1}$$

Degrees

- How can we compute these trigonometric functions?
- It is no good drawing a triangle with a given angle and trying to measure the lengths: an algorithm is needed.
- There is a good algorithm but there is a catch: using ‘degrees’ to measure angles is no good.
- It is fairly easy to see why degrees are flawed. There is nothing special about the number 360 for the number of degrees in a complete circle; other cultures have used different numbers. What is needed is a scale for measuring angles that is not arbitrary.

Radians



- We can measure the angle ϕ in radians as the ratio between the arc length round the circle a and the radius r.
- This does not depend on the size of the circle and there is no arbitrary choice of scale.
- One complete revolution of the circle sweeps out an arc length of $2\pi r$, so the corresponding angle is 2π.

Conversion

- Angles measured in radians are nearly always left in terms of multiples of π. Since π is an irrational number, there is no simple exact form.
- To convert between degrees and radians, we use the fact that 360° is the same as 2π radians, or equivalently that 180° is the same as π radians.
- Hence, to convert degrees to radians, multiply by $\pi/180$ and to convert radians to degrees, multiply by $180/\pi$.
- For example, 90° in radians is

$$90^\circ = 90 \times \frac{\pi}{180}\,\text{rad} = \frac{\pi}{2}\,\text{rad}\,.$$

Exercise

1. Express 45° in radians.

2. Express $5\pi/6$ radians in degrees.

Solution

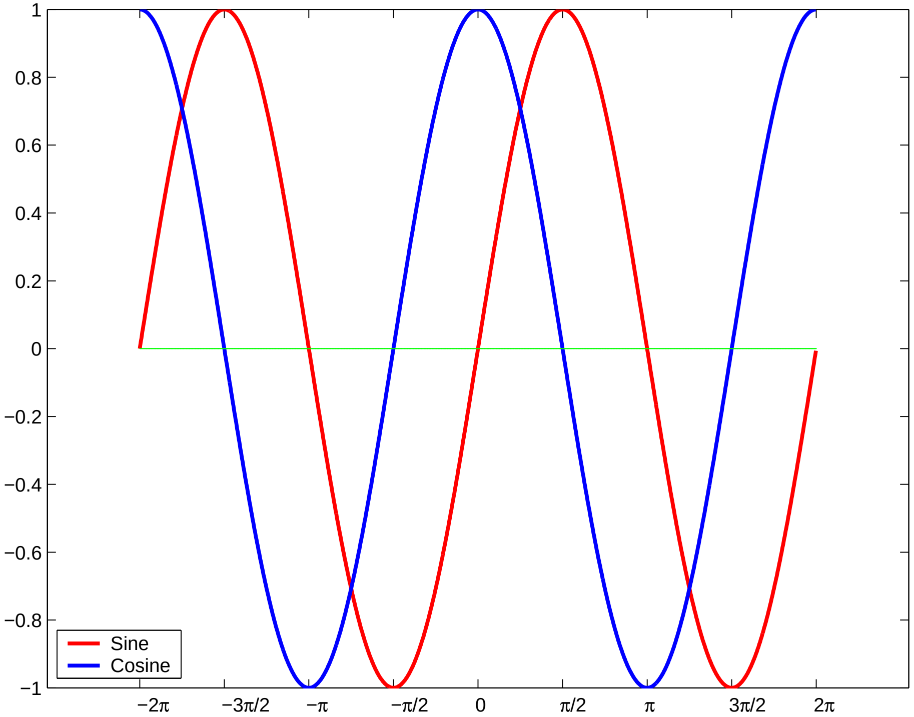
1. Express 45° in radians.

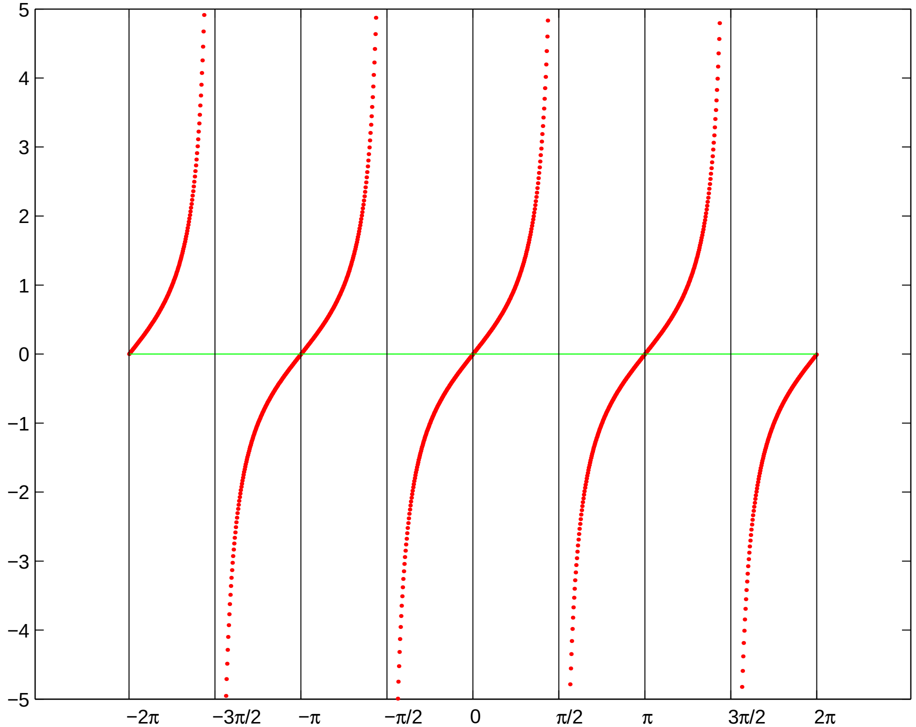
$$45^\circ = 45 \times \frac{\pi}{180}\,\text{rad} = \frac{\pi}{4}\,\text{rad}\,.$$

2. Express $5\pi/6$ radians in degrees.

$$\frac{5\pi}{6}\,\text{rad} = \frac{5\pi}{6} \times \frac{180}{\pi} = 5 \times 30 = 150^\circ.$$

Graphs


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Sine
Cosine
−2π
−3π/2
−π
−π/2
0
π/2
π
3π/2
2π


5
4
3
2
1
0
−1
−2
−3
−4
−5
−2π
−3π/2
−π
−π/2
0
π/2
π
3π/2
2π

Trigonometry in Ada

- Defined in the package `Ada.Numerics.Elementary_Functions`.

```
function Sin(X:          Float_Type) return Float_Type;
function Sin(X, Cycle:   Float_Type) return Float_Type;
function Cos(X:          Float_Type) return Float_Type;
function Cos(X, Cycle:   Float_Type) return Float_Type;
function Tan(X:          Float_Type) return Float_Type;
function Tan(X, Cycle:   Float_Type) return Float_Type;
```

  Each function comes in a pair with one or two arguments.

- The single–argument versions assume that the parameter is given in radians.
- In the two–argument versions, the second parameter gives the number of units in a whole cycle (i.e. a complete revolution). Thus, to find the sine of 30 degrees, we write

```
Sin(30.0, 360.0)
```

Powers

- The nth power of a number $b > 0$, written b^n is defined by:

$$b^n = \begin{cases} 1 & n = 0 \\ b & n = 1 \\ b \times b^{n-1} & n > 1 \end{cases}$$

  This means that b^n is the product of n copies of b; so $2^3 = 2 \times 2 \times 2$.

- We see powers of 2 a lot in computing: it is useful to remember that $2^{10} = 1024 \approx 1000$.

Logarithms

- The logarithm of a number y to a base $b > 0$, written $\log_b y$ is the number of copies of b that must be multiplied together to give y. So

$$\log_b(b^n) = n.$$

  For example, $\log_2(8) = 3$.

- The logarithm of an integer is usually a real number, not an integer.

Exercise

Evaluate the following.

1. 2^4, 2^8, 10^3.

2. $\log_2(32)$, $\log_2(1024)$, $\log_{10}(10000)$.

Solution

1.

$$2^4 = 16 \qquad 2^8 = 256 \qquad 10^3 = 1000.$$

2.

$$\log_2(32) = 5 \qquad \log_2(1024) = 10 \qquad \log_{10}(10000) = 4.$$

Properties of the Power Function

1.

$$b^m \times b^n = b^{m+n}. \tag{2}$$

Multiplying two numbers that are powers is the same as adding the powers. For example $2^3 \times 2^2 = 8 \times 4 = 32 = 2^5 = 2^{3+2}$.

2.

$$(b^m)^n = b^{mn}. \tag{3}$$

Raising one power to another power is the same as multiplying the powers. For example, $(2^3)^2 = 8^2 = 64 = 2^6 = 2^{2\times 3}$.

Consequences

- If $n \in \mathbb{Z}$ and $n < 0$, then

$$b^n = \frac{1}{b^n}.$$

  This is because $b^n \times b^{-n} = b^{n-n} = b^0 = 1$ by (2).

- If $n \in \mathbb{N}$, then $b^{1/n}$ is the nth root of b. For example, $b^{1/2} = \sqrt{b}$. This is because $(b^{1/n})^n = b^{n \times 1/n} = b^1 = b$ by (3).

- The identities (2) and (3) imply that we can compute b^n for all $n \in \mathbb{Q}$.

-

$$\log_b(mn) = \log_b(m) + \log_b(n). \tag{4}$$

  This identity follows by taking logs of both sides of (2).

$$\log_b(m^n) = n \times \log_b(m). \tag{5}$$

Take logs of both sides of (3).

$$a^m = b^{m \log_b a}. \tag{6}$$

Follows from (5) and the fact that raising to a power and taking logarithms are inverse functions. If we can raise one number b to any power, we can raise any other number, a, to any power. This means that we only need to find one algorithm for exponentiation, provided we can compute logs.

$$\log_b a = \frac{1}{\log_a b} \tag{7}$$

$$\log_b x = \frac{\log_a x}{\log_a b} \tag{8}$$

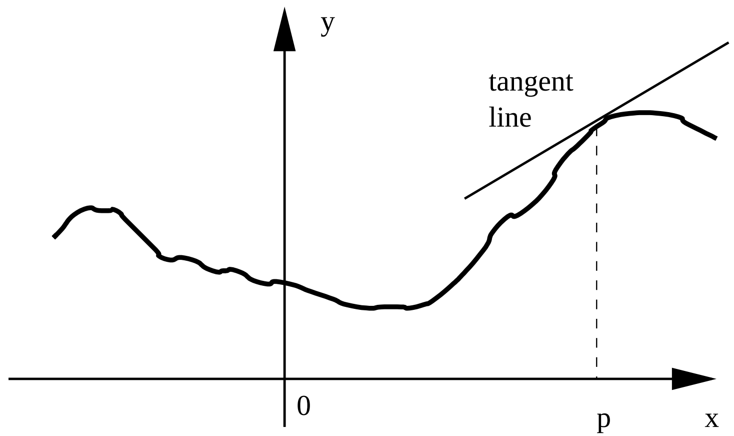
These identities follow from (6) with a little work. They imply that we only need to find one algorithm for calculating logarithms.

Remaining Problems

- Although we can define b^n for $n \in \mathbb{Q}$, we can't yet define it for irrational $n \in \mathbb{R}$.
- Computing b^n for $n \notin \mathbb{Z}$ involves finding roots, which cannot be done with the basic arithmetic operations.
- Computing $\log_b(a)$ is very difficult: with the definition, it means guessing a value x, computing b^x, comparing this with a, and then improving x. This is not a viable algorithm.

Choosing a Base for Logarithms

- Just as for trigonometric functions, there is one particular choice that is canonical; it is based on the fundamental structure of the function.
- Consider the rate of change of the function b^x with respect to x.
- It is quite common when modelling physical systems that functions are defined by their rate of change.



The rate of change of a function at a point x is equal to the gradient of the tangent at p.

Canonical Definition

- The power function $f(x) = b^x$ has the property that its rate of change with respect to x is equal to a constant multiple times the function: $kf(x)$.
- For a particular choice of the base, this constant k is equal to one. Unfortunately, this base is not a simple number; it is known as e and is equal to $\lim_{n\to\infty}(1 + 1/n)^n \approx 2.781828\ldots$.
- The 'exponential function' is e^x and logarithms to the base of e are known as natural logarithms and are often written $\ln x$.
- The term $\log x$ is ambiguous: to a mathematician, it means the natural logarithm, while to engineers it means log to base 10 $\log_{10}$.

Ada Functions

- There are three related Ada functions:

```
function Exp(X:             Float_Type) return Float_Type;
function Log(X:             Float_Type) return Float_Type;
function Log(X, Base:       Float_Type) return Float_Type;
```

- The single–parameter version of `Log` computes the natural logarithm.
- The two–parameter version of `Log` allows us to choose the base. For example, to find $\log_{10} 2$, write

```
Log(2.0, 10.0)
```

- We must have `X>0.0`, `Base>0.0`, and `Base/=1.0`

1. Elementary functions involve only arithmetic operations, trigonometric functions and their inverses, and logarithms and exponentiation.

2. The trigonometric functions sine, cosine and tangent are defined in terms of a rotating rod.

3. Degrees are an arbitrary scale for measuring angles. Radians are defined as the ratio between arc length and radius and are canonical.

4. Two important properties of the power function: $b^m \times b^n = b^{m+n}$ and $(b^m)^n = b^{mn}$.

5. Two important properties of the log function: $\log_b(mn) = \log_b(m) + \log_b(n)$ and $\log_b(m^n) = n \times \log_b(m)$.

6. The canonical base for logarithms is the number $e = \lim_{n\to\infty}(1 + 1/n)^n \approx 2.781828\ldots$.

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