Elementary Functions and Graphs

Three large topics in the next three lectures:

Graph sketching: getting a picture of simple functions.

Elementary functions: beyond arithmetic. Trigonometric, logarithmic, exponential, polynomial and rational functions.

Computing Elementary functions: sequences, series and recurrences.

Just skim the surface: differential calculus is essential to really understand these topics.

Why Study Elementary Functions?

Modelling the world. Elementary functions arise naturally as a way of describing the world around us:

- Trigonometric functions are fundamental to geometry and this is essential for computer graphics and gaming.
- Exponential functions occur as natural models for decay (e.g. sound in a room), arrival times in queues (e.g. for modelling call centre services, traffic, processing plants), and reliability (e.g. time to failure for components).

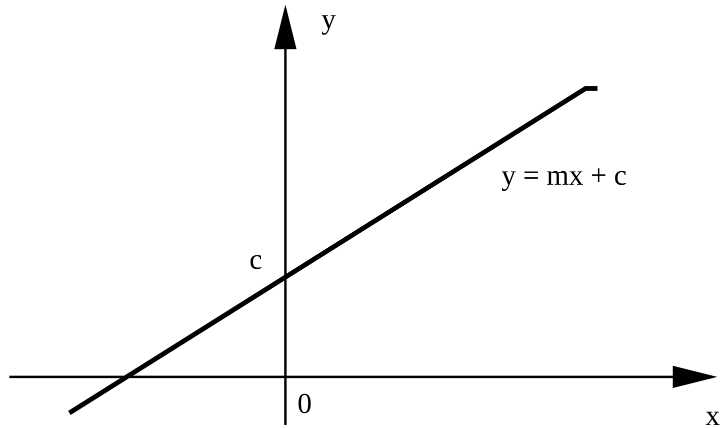
Understanding computational complexity. Algorithmic complexity is the study of how long algorithms take to run as a function of problem size (e.g. the length of the list). Exponential and logarithmic functions frequently turn up as measures of algorithmic complexity.

Graphs

- The ability to produce a picture of a problem is a useful tool for solving it.
- From the graph of a function $y = f(x)$ we are able to read off such things as the number of solutions to the equation $f(x) = 0$, regions over which it is increasing or decreasing, and the points where it is not defined.
- In order to draw the graphs of a large number of functions, we need only remember a few key graphs and be able to apply some simple techniques for transformations.
- A sketch of a graph is one which is not drawn strictly to scale but shows its most important features.

Straight Lines

- The function $y = mx + c$ is said to be linear because its graph is a straight line.
- x is the input variable and there are two parameters: m is the gradient or slope, and c is the value of y when $x = 0$, also known as the intercept.
- What does the graph look like if $m = 0$?



Sketching a Straight Line

A straight line is uniquely defined by any two distinct points that lie on it (this is because there are two parameters). The quickest way to sketch a straight line graph is to find the points where it crosses the axes.

To sketch the graph of $y = 4x - 2$:

1. To find where the graph crosses the y-axis, substitute $x = 0$ into the equation of the line:

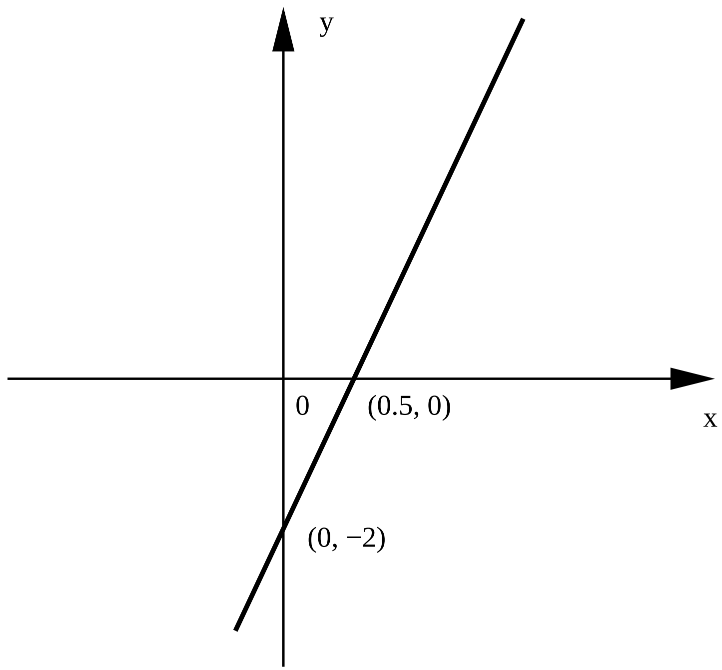
$$y = 4 \times 0 - 2 = -2.$$

   This means that the graph passes through the point $(0, -2)$.

2. To find where the graph crosses the x-axis, substitute $y = 0$ into the equation:

$$0 = 4x - 2 \Leftrightarrow 4x = 2 \Leftrightarrow x = \frac{2}{4} = 0.5.$$

Sketch of $y = 4x - 2$



Sketch $y = -0.5x - 1$.

Quadratic Functions

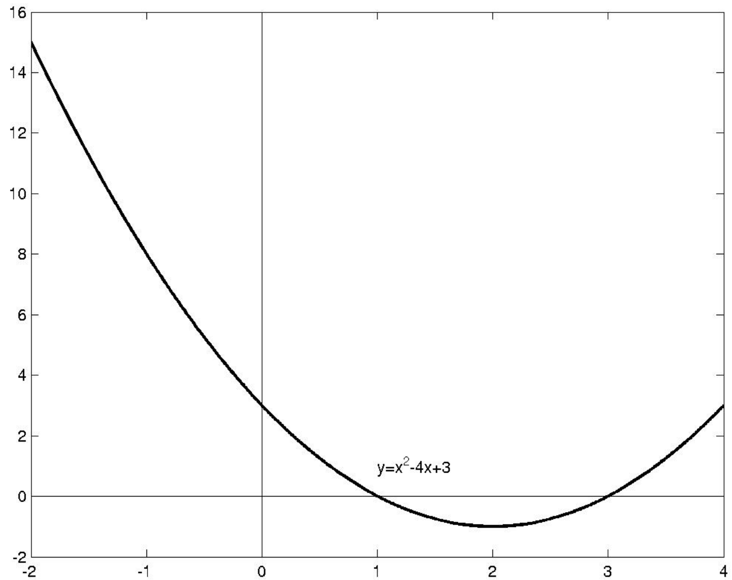
- The equation $y = ax^2 + bx + c$ is a general way of writing a function in which the highest power of x is a squared term.
- This is called a quadratic function, and its graph is called a parabola.
- Neglecting the effect of air resistance and the earth's curvature, projectiles (objects that are propelled and then left to move) follow a parabolic curve.

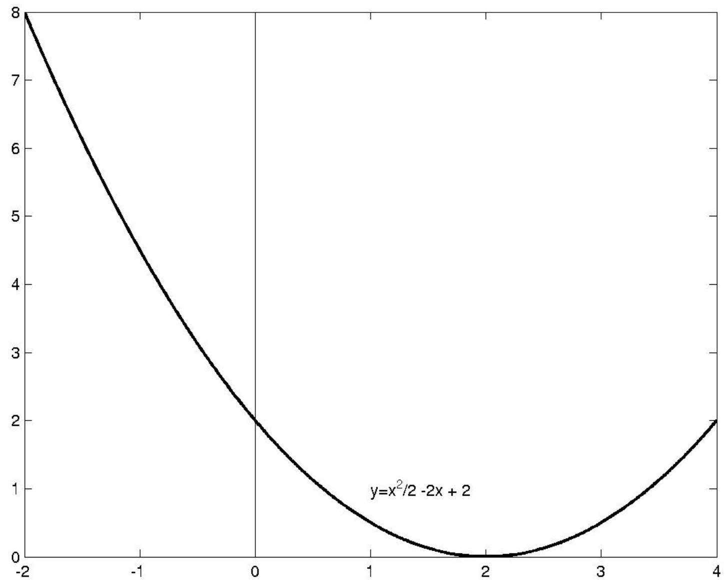
Sketching a Quadratic

- To sketch a quadratic we work out where it crosses the x- and y-axes.
- The y-axis is simple; the point $(0, c)$.
- To find where the graph crosses the x-axis we compute the rootsof the equation, $f(x) = 0$, using a formula:

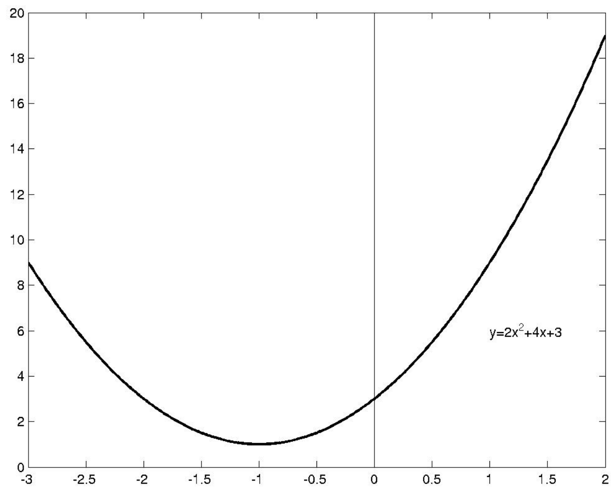
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

- The term $b^2 - 4ac$ is called the discriminant of the function.





Discriminant > 0; two real roots. Discriminant $= 0$; one real root.



Discriminant < 0; no real roots.

Exercise

Calculate the discriminant for the three quadratics:

1. $y = x^2 - 4x + 3$;
2. $y = \frac{1}{2}x^2 - 2x + 2$;
3. $y = 2x^2 + 4x + 3$.

Do the results agree with the sketches?

Solution

Calculate the discriminant for the three quadratics:

1. $y = x^2 - 4x + 3$: 4.
2. $y = \frac{1}{2}x^2 - 2x + 2$: 0.
3. $y = 2x^2 + 4x + 3$: -8.

Growth Rates

- Dor large values of x, a quadratic function increases very quickly, much faster than a linear function.
- In fact, the quadratic function is dominated by the coefficient of x^2; if this is negative, then $f(x)$ goes to $-\infty$ as x increases.
- Quite often, we are more interested in the behaviour of functions as x is large rather than the intricate details of what happens when x is small. For example, if the time taken by one algorithm for sorting is linear (in the number of items), while another is quadratic, we can conclude that the linear algorithm will be better for all large lists, because for large enough x, the quadratic function will be larger than the linear one.

Transformations

- Translation $g(x) = f(x + a)$. The graph of $g(x)$ is found by translating the graph of $f(x)$ to the left by a units.
- Translation $g(x) = f(x) + a$. The graph of $g(x)$ is found by translating the graph of $f(x)$ up by a units.
- Reflection $g(x) = f(-x)$. The graph of $g(x)$ is found by reflecting the graph of $f(x)$ in the y-axis.
- Reflection $g(x) = -f(x)$. The graph of $g(x)$ is found by reflecting the graph of $f(x)$ in the x-axis.

More Transformations

- Scaling $g(x) = f(ax)$. The graph of $g(x)$ is found by rescaling the x-axis. The graph is squashed horizontally if $a > 1$ and is stretched horizontally if $0 < a < 1$.
- Scaling $g(x) = af(x)$. The graph of $g(x)$ is found by rescaling the y-axis. The graph is stretched vertically if $a > 1$ and is squashed vertically if $0 < a < 1$.
- Reflecting in the line $y = x$. This has the effect of swapping x and y in the definition of the function, and gives the graph of the inverse function.

Exercise

Draw a sketch of $y = 2x + 1$. Using standard transformations, draw $y = 2x + 4$, $y = -2x + 1$, $y = -2x - 1$.

Summary

1. A sketch of a graph is one which is not drawn strictly to scale but shows its most important features.
2. A straight line is uniquely defined by any two distinct points that lie on it. The two easiest points to choose are where it crosses the axes.
3. A parabola is the shape of the graph of a quadratic function.
4. To find where a parabola crosses the x-axis we must find its roots using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

5. Knowing the effect of standard transformations can make it much easier to sketch a function.