Session Objectives

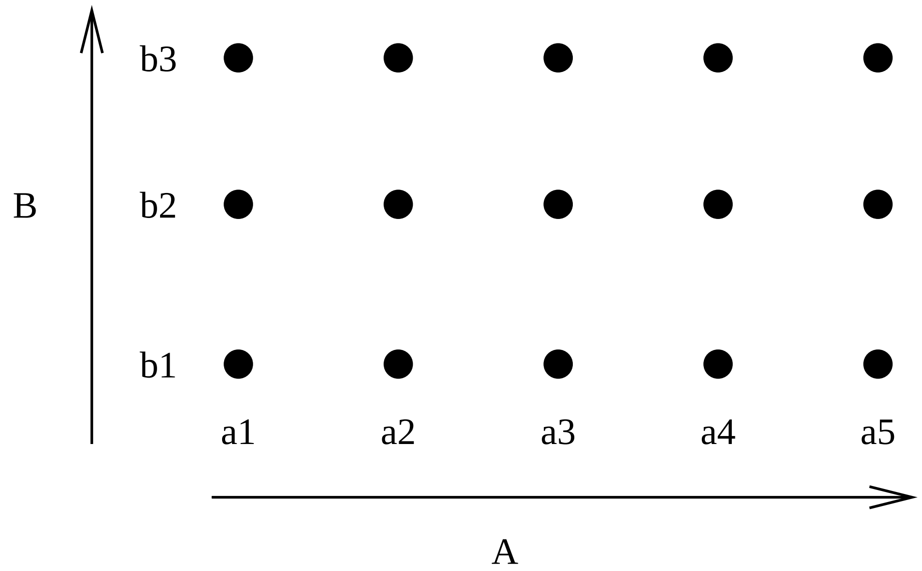
- Define Cartesian products and ordered pairs
- Define relations
- Compute the domain and range of a relation
- Check if a relation is symmetric, reflexive and transitive
- Check if a relation is injective, surjective or bijective
- Define functions and their inverses

Cartesian Products and Ordered Pairs

- The Cartesian product $A \times B$ is a set consisting of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
- Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.
- If $A = \emptyset$ or $B = \emptyset$, then we define $A \times B = \emptyset$.
- It is easy to show that $|A \times B| = |A| \times |B|$.
- The Cartesian product of a set A with itself can be written as A^2.

Grid View

- Geometrically, we can view $A \times B$ as a rectangular grid. We can call this the grid view.
- The x-coordinates are given by elements of A and the y-coordinates are given by elements of B.



Examples

1. Let $A = \{x, y\}$ and $B = \{p, q, r\}$. Then

$$A \times B = \{(x, p), (x, q), (x, r), (y, p), (y, q), (y, r)\}.$$

Note that $|A \times B| = 6 = 2 \times 3 = |A| \times |B|$.

2. If $A = \{0, 1, 2\}$ and $B = \{5, 6\}$, what is $A \times B$?

3. The set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the Cartesian product of $\mathbb{R}$ with itself, and also the Cartesian plane. We can also see how $\mathbb{R}^n$ is defined in set theoretic terms.

Solution

If $A = \{0, 1, 2\}$ and $B = \{5, 6\}$, what is $A \times B$?

Solution:

$$A \times B = \{(0, 5), (0, 6), (1, 5), (1, 6), (2, 5), (2, 6)\}.$$

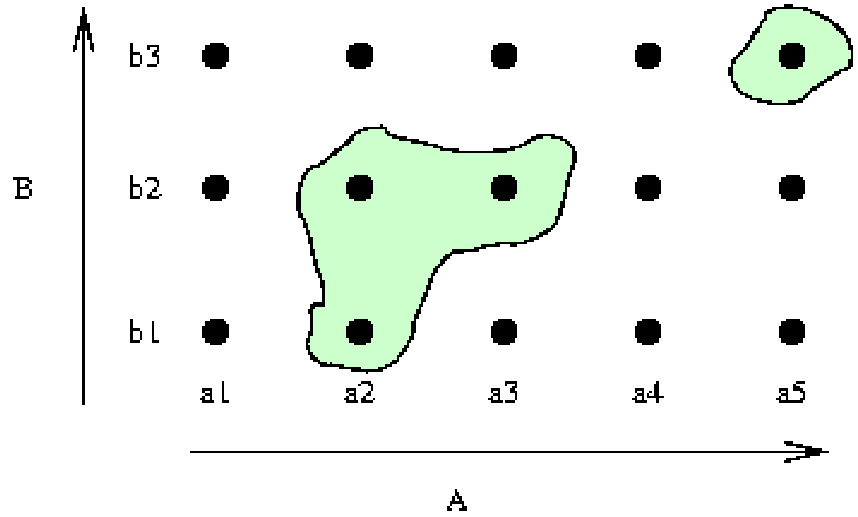
Note that $|A \times B| = 6 = |A| \times |B|$. This is a useful check to see if we have correctly listed all the elements.

Relations

- A relation is the mathematical way of describing a given relationship between elements of two sets. For example, if A and B are two sets of people, the 'parent' relation would define which members of B were parents of members of A.
- A binary relation from A to B is a set R of ordered pairs; that is $R \subseteq A \times B$. In the grid view, it is just a region of the rectangular grid.
- The set R is defined so that it includes the elements that we think are relevant to a particular relation.
- If an ordered pair are related, so $(a, b) \in R$, then we can also write this aRb.

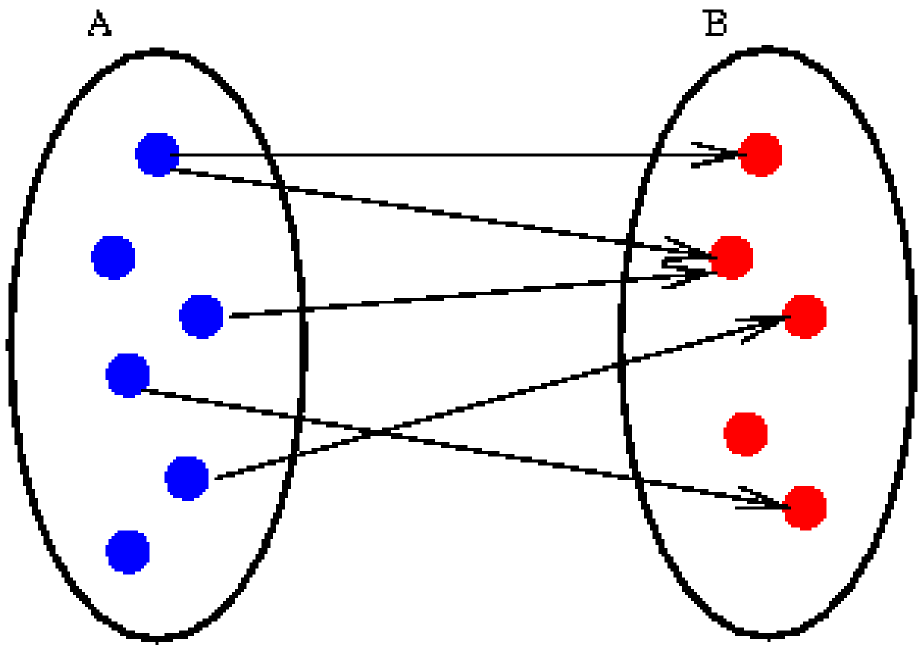
Grid View

- Draw the grid.
- Draw a region that includes the correct points.

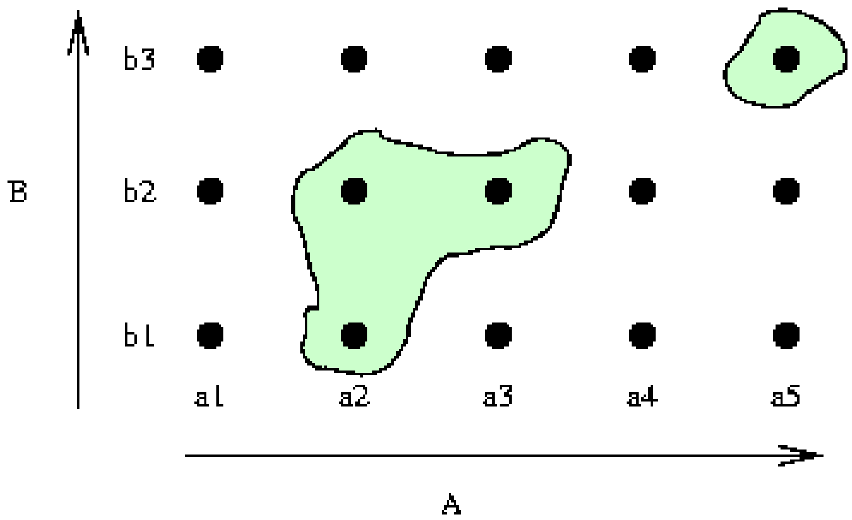


Geometric View

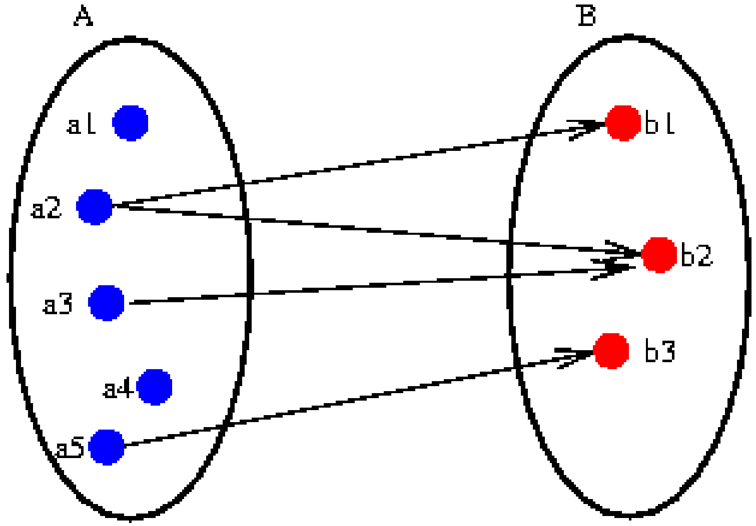
- Draw the sets A and B as ellipses.
- Join a point $a \in A$ to a point $b \in B$ by an arrow if aRb.



Comparison



Grid View.



Geometric View.

Examples

1. If $A = \{0, 1, 2\}$ and $B = \{5, 6\}$ then we can define a relation $R = \{(1, 5), (0, 6)\}$. We say that $1R5$ and $0R6$. Another relation S could be defined by $S = \{(2, 5)\}$, so $2S5$ are the only two elements related by S.
2. If A is the set of Aston students and B is the set of books owned by the University library, then a relation R can be defined by aRb if and only if, student a has book b on loan.
3. Let A be a set. Then there is an identity relation R on A with the property that $a_1 R a_2$ if and only if $a_1 = a_2$. We can define R as a subset of $A \times A$ as follows:

$$R = \{(a, a) \mid a \in A\}.$$

Domain of a Relation

- We define the domain of a relation R from A to B as the set of elements of A that are related to some element of B:

$$\text{dom}(R) = \{a \in A \mid \exists b \in B, (a, b) \in R\}. \quad (1)$$

  This definition can be read as 'the set of elements $a \in A$ such that there exists an element $b \in B$ with the property that aRb'.

- In the grid view, we project all the points that belong to the relation down to the A-axis.

- In the geometric view, we find all the points $a \in A$ that an arrow starts from.

Range of a Relation

- We define the range or co-domain of a relation A from A to B as the set of elements of B that are related to some element of A:

$$\text{ran}(R) = \{b \in B \mid \exists a \in A, (a, b) \in R\}. \tag{2}$$

- In the grid view, we project all the points that belong to the relation across to the B-axis.
- In the geometric view, we find all the points $b \in B$ that an arrow ends at.

Examples

1. If $A = \{0, 1, 2\}$ and $B = \{5, 6\}$ then we can define a relation $R = \{(1, 5), (0, 6)\}$.

$$\operatorname{dom} R = \{0, 1\} \subset A \qquad \operatorname{ran} R = \{5, 6\} \subset B.$$

2. If A is the set of Aston students and B is the set of books owned by the University library, then a relation R can be defined by aRb if and only if, student a has book b on loan.

   $\operatorname{dom} R$ is the set of students who are borrowing a book; $\operatorname{ran} R$ is the set of books which are currently on loan.

3. Let A be a set. Then there is always an identity relation R on A with the property that $a_1 R a_2$ if and only if $a_1 = a_2$.

$$\operatorname{dom} R = A \qquad \operatorname{ran} R = A.$$

Inverse Relations

- The inverse R^{-1} of a relation R is found simply by reversing the ordered pairs that define R.

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}. \tag{3}$$

  The same pairs of elements are related, we just view the relation in the opposite direction.

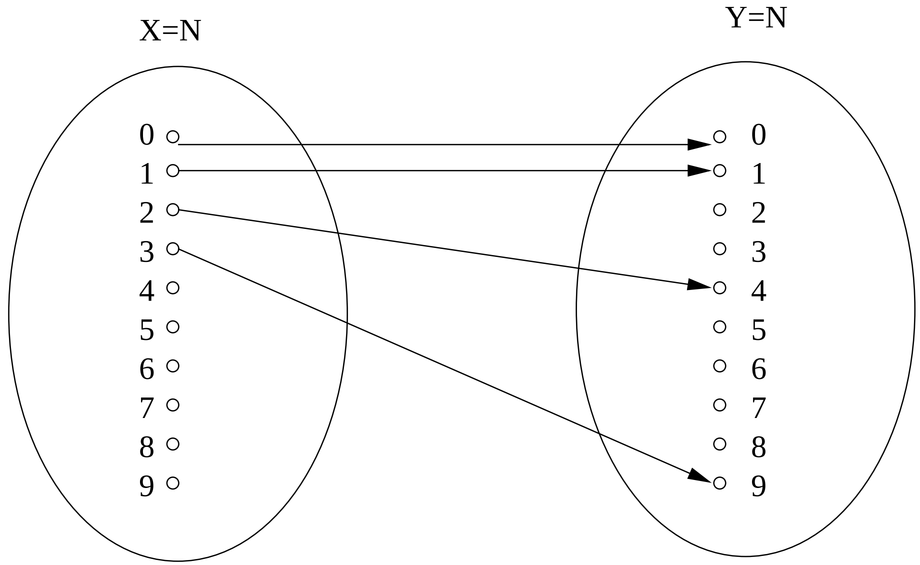
- In the grid view, we swap the axes over. In the geometric view, we reverse the order of the arrows.
- The inverse of the relation 'is a parent of' is the relation 'is a child of'.
- The inverse of the relation $\leq$ is the relation $\geq$.
- The inverse of the identity relation is itself.

Functions

- A function is defined as a special sort of binary relation, namely one for which no member of the domain occurs more than once as a first coordinate. Equivalently, R is a function if $(x, y) \in R$ and $(x, z) \in R$ imply that $y = z$.
- This means that a function defines a rule as follows: if $a \in \operatorname{dom} R$ then there must be a unique b such that $(a, b) \in R$, and this b is the result of applying the rule to a.
- Although a function defines a rule for working out $f(a)$ for every $a \in \operatorname{dom} f$, it does not define an algorithm for calculating $f(a)$. In the mathematical definition, we simply know that some $f(a)$ exists.
- Although functions are defined in terms of ordered pairs, we generally use the notation $f(a) = b$ to denote $(a, b) \in R$. If the relation $R \subseteq X \times Y$ we write $f : X \to Y$.

Example

- Note that we do not require that the range of f to be the whole of Y.
- We may not be able to work out what that range is for perfectly good functions.
- The function $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(m) = m^2$ does not have the whole of $\mathbb{N}$ as its range.



Function Properties: Injective

- f is injective (or one-to-one) if each $y \in \operatorname{ran} f$ is the image of only one $x \in X$. This means that if $x_1 \neq x_2 \in \operatorname{dom} f$ then $f(x) \neq f(y)$.
- In the grid view, a function is injective if every horizontal slice through the function has at most one point in it.
- In the geometric view, a function is injective if every point in the range has at most one arrow coming into it.

Function Properties: Surjective

- f is surjective (or onto) if every $y \in Y$ is the image of an element $x \in X$. This means that $\operatorname{ran} f = Y$.
- In the grid view, a function is surjective if every horizontal slice through the function has at least one point in it.
- In the geometric view, a function is surjective if every point in Y is at the end of an arrow.

A function that is both injective and surjective is said to be bijective.

Examples

1. The relation xRy defined as "y is the mother of x" is a function, since every person has just one mother. In general, this function $m(x)$ is not injective, since a mother can have more than one child (i.e. we could have
$m(\text{George Weasley}) = m(\text{Fred Weasley})$); it would be injective if the domain were defined to be the set of only children. If the target set Y is the set of all people, then m is not surjective, since there are many people who aren't mothers.
2. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$ is injective, since $m^2 = n^2 \Rightarrow m = n$ for $m, n \in \mathbb{N}$. The function is not surjective, since many natural numbers are not squares.
3. The function $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ defined by $f(x) = x^2$ is not injective, since $x^2 = (-x)^2$. For example, $(-1)^2 = 1^2$. However, it is surjective, since $\mathbb{R}^+ \cup \{0\}$ denotes the non-negative real numbers, and every such number is the square of another real number.

Exercise

Determine which of the following relations are functions. For those which are, find the domain and the range, and whether they are one-to-one.

1. $\{(a, a), (a, b), (a, c)\}$
2. $\{(a, a), (b, a), (c, a)\}$
3. $\{(W, L) \mid W$ is a word in English with last letter $L\}$ Here L denotes any lower case letter.
4. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Solution

1. $\{(a, a), (a, b), (a, c)\}$ is not a function.
2. $\{(a, a), (b, a), (c, a)\}$ is a function. Domain is $\{a, b, c\}$ and range is $\{a\}$. It is not injective.
3. $\{(W, L) \mid W$ is a word in English with last letter $L\}$ Here L denotes any lower case letter. is a function. domain is set of words, range is whole alphabet (e.g. "quiz", "box", "Iraq"). It is not injective.
4. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is not a function (e.g. $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$).

Function Inverses

- The inverse f^{-1} of a function f is a function if and only if f is injective. The inverse has the property that $f^{-1} \circ f = \text{id}$, where id denotes the identity function.
- The domain of f^{-1} is the range of f. Thus if $f : X \to Y$, $f^{-1} : Y \to X$ if and only if f is injective and surjective (i.e., is bijective).

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is bijective. Its inverse is given by $f^{-1}(y) = (y - 3)/2$.

Injective suppose that $f(x_1) = f(x_2)$:

$$f(x_1) = f(x_2) \Leftrightarrow 2x_1 + 3 = 2x_2 + 3$$

$$\Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2$$

Surjective Consider an arbitrary $y \in \mathbb{R}$. If $y = 2x + 3$, then we can rewrite this to compute x in terms of y:

$$y = 2x + 3 \Leftrightarrow y - 3 = 2x$$

$$\Leftrightarrow \frac{y - 3}{2} = x$$

This shows that $f^{-1}(y) = (y - 3)/2$.

Summary

1. The Cartesian product of two sets A and B is a set consisting of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
2. Objects whose type is a Cartesian product are modelled naturally by records in Ada.
3. A binary relation from A to B is a set R of ordered pairs; that is $R \subseteq A \times B$.
4. A relation may be reflexive, symmetric or transitive. If it is all three it is known as an equivalence relation.
5. The tables in a relational database define a relation.
6. A function is a relation where no element appears more than once as the first term in an ordered pair.
7. Functions may be injective (one-to-one) or surjective (onto). A function that is both is said to be bijective.
8. A function has an inverse if and only if it is injective.

Session Objectives

- Define Cartesian products and ordered pairs
- Define relations
- Compute the domain and range of a relation
- Check if a relation is symmetric, reflexive and transitive
- Check if a relation is injective, surjective or bijective
- Define functions and their inverses