Session Objectives

- Define Cartesian products and ordered pairs
- Define relations
- Compute the domain and range of a relation
- Check if a relation is symmetric, reflexive and transitive
- Check if a relation is injective, surjective or bijective
- Define functions and their inverses

Cartesian Products and Ordered Pairs

- The Cartesian product $A \times B$ is a set consisting of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
- Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal if and only if $a_1 = a_2$ and $b_1 = b_2$. (This rather intuitive definition of an ordered pair can be made rigorous using a set theoretic construction: $(x, y) = \{x, \{x, y\}\}$).
- Note: an ordered pair is very different from a set. For example, $(0, 1) \neq (1, 0)$ but $\{0, 1\} = \{1, 0\}$.
- If $A = \emptyset$ or $B = \emptyset$, then we define $A \times B = \emptyset$.
- It is easy to show that $|A \times B| = |A| \times |B|$.

Higher Order Products

- The Cartesian product of a set A with itself can be written as A^2.
- We can extend the definition of Cartesian products to three or more sets: for example, $A \times B \times C$ is defined to be the set of all ordered triples (a, b, c) where $a \in A$, $b \in B$ and $c \in C$.
- An element of A^n has the form $(a_1, a_2, \ldots, a_n)$ where each $a_i \in A$: this is called an n-tuple.
- It can also be viewed as a list of size n.

Examples

1. Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$. Then

   $$A \times B = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\}.$$

   Note that $|A \times B| = 6 = 2 \times 3 = |A| \times |B|$.

2. If $A = \{a, b\}$ and $B = \{a, b, c\}$, what is $A \times B$?

3. The set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the Cartesian product of $\mathbb{R}$ with itself, and also the Cartesian plane. We can also see how $\mathbb{R}^n$ is defined in set theoretic terms.

Solution

If $A = \{a, b\}$ and $B = \{a, b, c\}$, what is $A \times B$?

Solution:

$$A \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c)\}.$$

Ada Examples

4. The Ada type declaration

```
TYPE   OrderType IS RECORD
       Quantity  : Natural;
       UnitPrice : Float;
END RECORD;
```

defines `OrderType` to be a Cartesian product type `Natural x Float`. Values of this type belong to the set $\mathbb{N} \times \mathbb{R}$.

5. Consider the following Ada type declaration:

```
TYPE   Vector3 IS ARRAY(1..3) OF Float;
```

What is the set that defines this type?

Relations

- A relation is the mathematical way of describing a relationship between elements of two sets. For example, if A and B are two sets of people, the 'parent' relation would define which members of B were parents of members of A.
- A binary relation from A to B is a set R of ordered pairs; that is $R \subseteq A \times B$. We can also define unary, ternary, ... relations in the obvious way; if we just refer to a 'relation' then we shall mean a binary relation.
- If an ordered pair $(a, b) \in R$ then we can also write aRb.

Examples

1. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then the relation $R = \{(1, 2), (2, 4)\}$ can be interpreted as aRb if and only if $b = 2a$.
2. If A is the set of Aston students and B is the set of books owned by the University library, then a relation R can be defined by aRb if and only if, student a has book b on loan.
3. Let A be a set. Then there is an identity relation R on A with the property that $a_1 R a_2$ if and only if $a_1 = a_2$. Write down a definition of R as a subset of $A \times A$.

Solution

Let A be a set. Then there is an identity relation R on A with the property that $a_1 R a_2$ if and only if $a_1 = a_2$. Write down a definition of R as a subset of $A \times A$.

Solution:

$$R = \{(a, a) \mid a \in A\}$$

Range and Domain

- We define the domain of a relation R from A to B as the set of elements of A that are related to some element of B:

$$\mathsf{dom}(R) = \{a \in A \mid \exists b \in B, (a, b) \in R\}. \tag{1}$$

  This definition can be read as 'the set of elements $a \in A$ such that there exists an element $b \in B$ with the property that aRb'.

- We define the range or co-domain of a relation A from A to B as the set of elements of B that are related to some element of A:

$$\mathsf{ran}(R) = \{b \in B \mid \exists a \in A, (a, b) \in R\}. \tag{2}$$

Write down the domain and range of the relations defined in the previous example.

Solution

1. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then the relation $R = \{(1, 2), (2, 4)\}$ can be interpreted as aRb if and only if $b = 2a$.

   $$\operatorname{dom} R = \{1, 2\} \subset A \qquad \operatorname{ran} R = \{2, 4\} \subset B.$$

2. If A is the set of Aston students and B is the set of books owned by the University library, then a relation R can be defined by aRb if and only if, student a has book b on loan.

   $\operatorname{dom} R$ is the set of students who are borrowing a book; $\operatorname{ran} R$ is the set of books which are currently on loan.

3. Let A be a set. Then there is always an identity relation R on A with the property that $a_1 R a_2$ if and only if $a_1 = a_2$.

   $$\operatorname{dom} R = A \qquad \operatorname{ran} R = A.$$

Inverse Relations

- The inverse R^{-1} of a relation R is found simply by reversing the ordered pairs that define R.

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}. \tag{3}$$

- The inverse of the relation 'is a parent of' is the relation 'is a child of'.
- The inverse of the relation $\leq$ is the relation $\geq$.
- The inverse of the identity relation is itself.

Equivalence Relations

Reflexivity R is reflexive if aRa for all $a \in A$.

Symmetry R is symmetric if aRb if and only if bRa. This is equivalent to requiring either both of (a, b) and (b, a) to be in R or neither of them.

Transitivity R is transitive if both aRb and bRc imply that aRc.

- A relation that is reflexive, symmetric and transitive is called an equivalence relation.
- We define the equivalence class of an element $a \in A$ to be the set $\{b \in A \mid (a, b) \in R\}$ (so this is a subset of A) and denote it by $[a]$.
- It can be shown that the set of equivalence classes form a partition of A; this means that the union of the classes covers the whole of A, while they are all disjoint.

Example: Ordering

On $\mathbb{R}$, the relation $\geq$ is reflexive and transitive, but not symmetric.

- The relation is reflexive because $x \geq x$ for all $x \in \mathbb{R}$ (this would not be true for the relation $>$).
- The relation is transitive because if $a \geq b$ and $b \geq c$, then it is clearly true that $a \geq c$.
- The relation is not symmetric because $2 \geq 1$ but $1 \not\geq 2$. Note that these properties must be true for all elements of a set for them to hold. It is not good enough for symmetry to note that $2 \geq 2$ (both ways round).

Modular Arithmetic

The modulo-n relation defined on the integers by mRn if and only if $x - y$ is divisible by n is an equivalence relation.

- For every integer x, $x - x = 0$ is divisible by n. The relation is reflexive.
- If $x - y$ is divisible by n, then so is $y - x$. (If $x - y = an$, for $a \in \mathbb{Z}$, then $y - x = (-a)n$). The relation is symmetric.
- If both $y - x$ and $z - y$ are divisible by n, then so is $z - x = (z - y) - (y - x)$. The relation is transitive.

The equivalence classes of this relation are $[0], [1], \ldots, [n - 1]$. Arithmetic operators can be defined on these equivalence classes in a natural way; for example,

$$[x] + [y] = [x + y].$$

Relations and Databases

- A table in a relational database stores information linking n attributes of a set of individuals as a family of n-tuples.
- For example, information about a student on this module might be stored as a quintuple

$$(\text{name}, \text{program}, \text{email address}, \text{coursework mark}, \text{exam mark})$$

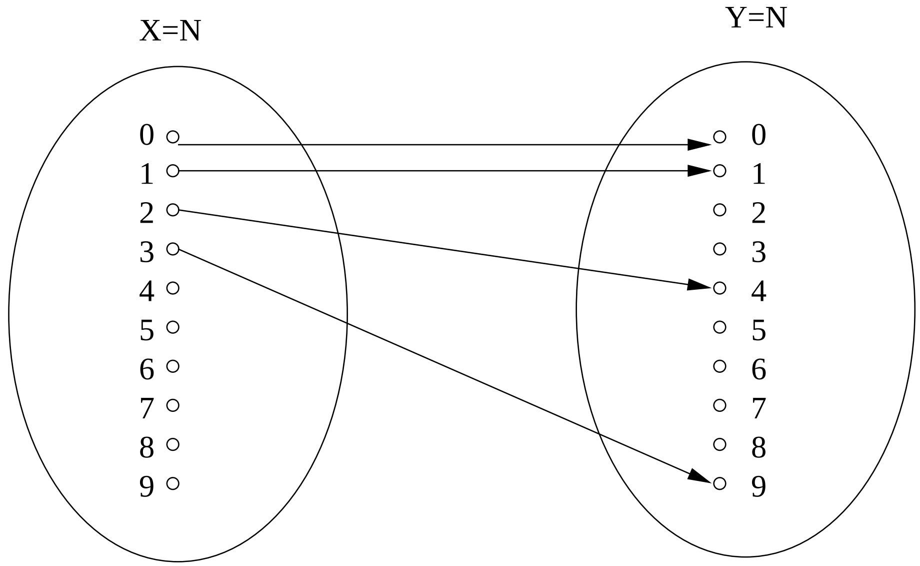
- Some of this information is common to other tables (for example, the information about students on other first year CS modules).
- Understanding the way in which relations are themselves related to each other is important in order to reorganise the information so that it can be read, updated, and searched efficiently. This is done through the classification of the various normal forms of relations.

Functions

- A function is defined as a special sort of binary relation, namely one for which no member of the domain occurs more than once as a first coordinate. Equivalently, R is a function if $(x, y) \in R$ and $(x, z) \in R$ imply that $y = z$.
- This means that a function defines a rule as follows: if $a \in \operatorname{dom} R$ then there must be a unique b such that $(a, b) \in R$, and this b is the result of applying the rule to a.
- Although a function defines a rule for working out $f(a)$ for every $a \in \operatorname{dom} f$, it does not define an algorithm for calculating $f(a)$. In the mathematical definition, we simply know that some $f(a)$ exists.
- Although functions are defined in terms of ordered pairs, we generally use the notation $f(a) = b$ to denote $(a, b) \in R$. If the relation $R \subseteq X \times Y$ we write $f : X \to Y$.

Example

- Note that we do not require that the range of f to be the whole of Y.
- We may not be able to work out what that range is for perfectly good functions.
- The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(m) = m^2$ does not have the whole of $\mathbb{N}$ as its range.



Function Properties

Injective f is injective (or one-to-one) if each $y \in \operatorname{ran} f$ is the image of only one $x \in X$. This means that if $x_1 \neq x_2 \in \operatorname{dom} f$ then $f(x) \neq f(y)$.

Surjective f is surjective (or onto*) if every $y \in Y$ is the image of an element $x \in X$. This means that $\operatorname{ran} f = Y$.

A function that is both injective and surjective is said to be bijective.

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*I hate this word: how can we use a preposition as an adjective. However, it is easier to remember its meaning.

Examples

1. The relation xRy defined as "y is the mother of x" is a function, since every person has just one mother. In general, this function $m(x)$ is not injective, since a mother can have more than one child (i.e. we could have $m(\text{Bill}) = m(\text{Fred})$); it would be injective if the domain were defined to be the set of only children. If the target set Y is the set of all people, then m is not surjective, since there are many people who aren't mothers.
2. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$ is injective, since $m^2 = n^2 \Rightarrow m = n$ for $m, n \in \mathbb{N}$. The function is not surjective, since many natural numbers are not squares.
3. The function $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ defined by $f(x) = x^2$ is not injective, since $x^2 = (-x)^2$. For example, $(-1)^2 = 1^2$. However, it is surjective, since $\mathbb{R}^+ \cup \{0\}$ denotes the non-negative real numbers, and every such number is the square of another real number.

Exercise

Determine which of the following relations are functions. For those which are, find the domain and the range, and whether they are one-to-one.

1. $\{(a, a), (a, b), (a, c)\}$
2. $\{(a, a), (b, a), (c, a)\}$
3. $\{(W, L) \mid W$ is a word in English with last letter $L\}$ Here L denotes any lower case letter.
4. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Solution

1. $\{(a,a),(a,b),(a,c)\}$ is not a function.
2. $\{(a,a),(b,a),(c,a)\}$ is a function. Domain is $\{a,b,c\}$ and range is $\{a\}$. It is not injective.
3. $\{(W,L) \mid W$ is a word in English with last letter $L\}$ Here L denotes any lower case letter. is a function. domain is set of words, range is whole alphabet (e.g. “quiz”, “box”, “Iraq”). It is not injective.
4. $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is not a function (e.g. $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$).

Composition of Relations

- For two relations R and S we form the composition $S \circ R$ by defining $(x, y) \in S \circ R$ if and only if there is some z such that $(x, z) \in R$ and $(z, y) \in S$.
- The confusing part of this definition is that although R is applied first, it is written on the right. This is to conform with the usual notation for functions.
- Let R be the relation 'is brother of', and S the relation 'is parent of'. Then $S \circ R$ is the relation 'is uncle of'. For $(x, y) \in S \circ R$ if and only if there is z such that x is a brother of z and z is a parent of y, that is x is an uncle of y.

Function Inverses

- The inverse f^{-1} of a function f is a function if and only if f is injective. The inverse has the property that $f^{-1} \circ f = \mathrm{id}$, where id denotes the identity function.
- The domain of f^{-1} is the range of f. Thus if $f : X \to Y$, $f^{-1} : Y \to X$ if and only if f is injective and surjective (i.e., is bijective).

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is bijective. Its inverse is given by $f^{-1}(y) = (y - 3)/2$.

Injective suppose that $f(x_1) = f(x_2)$:

$$
\begin{aligned}
f(x_1) = f(x_2) &\Leftrightarrow 2x_1 + 3 = 2x_2 + 3 \\
&\Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2
\end{aligned}
$$

Surjective Consider an arbitrary $y \in \mathbb{R}$. If $y = 2x + 3$, then we can rewrite this to compute x in terms of y:

$$
\begin{aligned}
y = 2x + 3 &\Leftrightarrow y - 3 = 2x \\
&\Leftrightarrow \frac{y-3}{2} = x
\end{aligned}
$$

This shows that $f^{-1}(y) = (y - 3)/2$.

Summary

1. The Cartesian product of two sets A and B is a set consisting of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
2. Objects whose type is a Cartesian product are modelled naturally by records in Ada.
3. A binary relation from A to B is a set R of ordered pairs; that is $R \subseteq A \times B$.
4. A relation may be reflexive, symmetric or transitive. If it is all three it is known as an equivalence relation.
5. The tables in a relational database define a relation.
6. A function is a relation where no element appears more than once as the first term in an ordered pair.
7. Functions may be injective (one-to-one) or surjective (onto). A function that is both is said to be bijective.
8. A function has an inverse if and only if it is injective.

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