Session Objectives

- Explain how matrices represent linear transformations
- Write down matrices for some common transformations

Matrices and Linear Transformations

- An important application of matrices is in computer graphics.
- We often need to transform an image (or an object in an image) by moving or distorting it. Such transformations are often linear and can be represented by matrices. We shall use two-dimensional transformations as examples in this section, but all the ideas work just as well in three (or even more) dimensions.
- A linear transformation in $\mathbb{R}^n$ is defined uniquely by its action on n linearly independent vectors: we can take the unit basis vectors which are zero except for a 1 in the ith coordinate. So in $\mathbb{R}^2$, $e_1 = [1, 0]^T]$ and $e_2 = [0, 1]^T$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \tag{1}$$

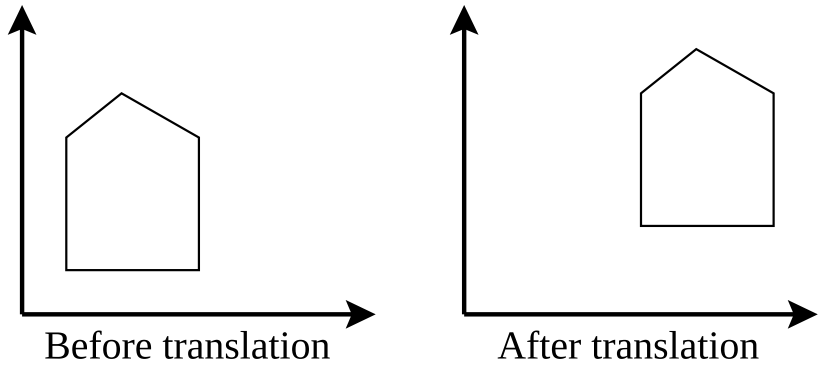
  This shows that the image of e_1 is the first column of the matrix, and the image of e_2 is the second column.

Translation

- Translations are simple movements or offsets from the existing position of the object.
- If we move each point of an image by a distance v_1 in the x-direction and v_2 in the y-direction, then a point $\mathbf{p}$ with co-ordinates $[p_1, p_2]$ is mapped to $\mathbf{p}'$:

$$\mathbf{p}' = \mathbf{p} + \mathbf{v}, \qquad (2)$$

  where $\mathbf{v} = [v_1, v_2]'$.
- Note how the shape and size of the object are preserved by the translation.

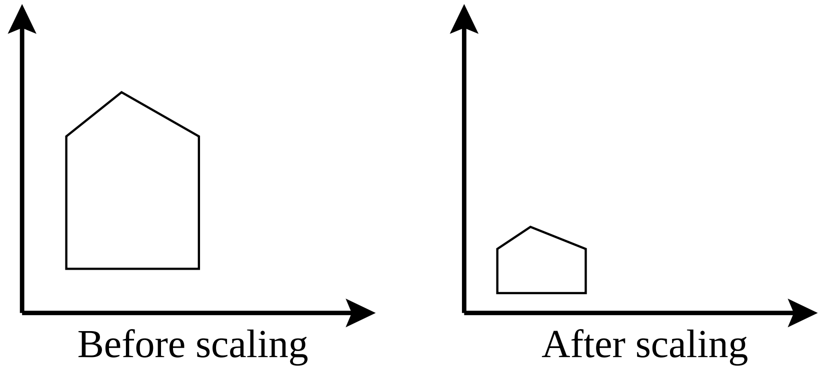


Scaling

- Scalings are simple stretchings of an object about the origin.
- If we rescale the x-axis by s_1 and the y-axis by s_2, then the transformation can be represented by the scaling matrix

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}.$$

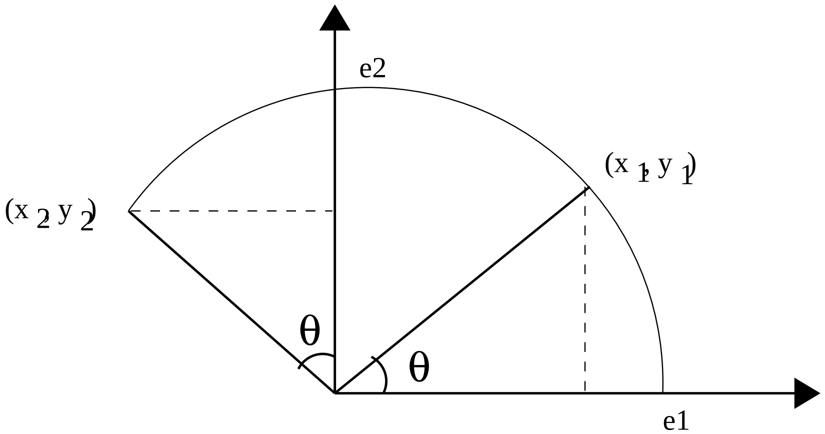
To compute the effect of the transformation, we multiply each point in the image by $\mathbf{S}$ so that $\mathbf{p}' = \mathbf{S}\mathbf{p}$.



If $s_x = s_y$ then the scaling is said to be uniform. The shape of the object is only preserved by uniform scaling.

Rotation

We define a rotation about the origin through an angle θ measured anti-clockwise.

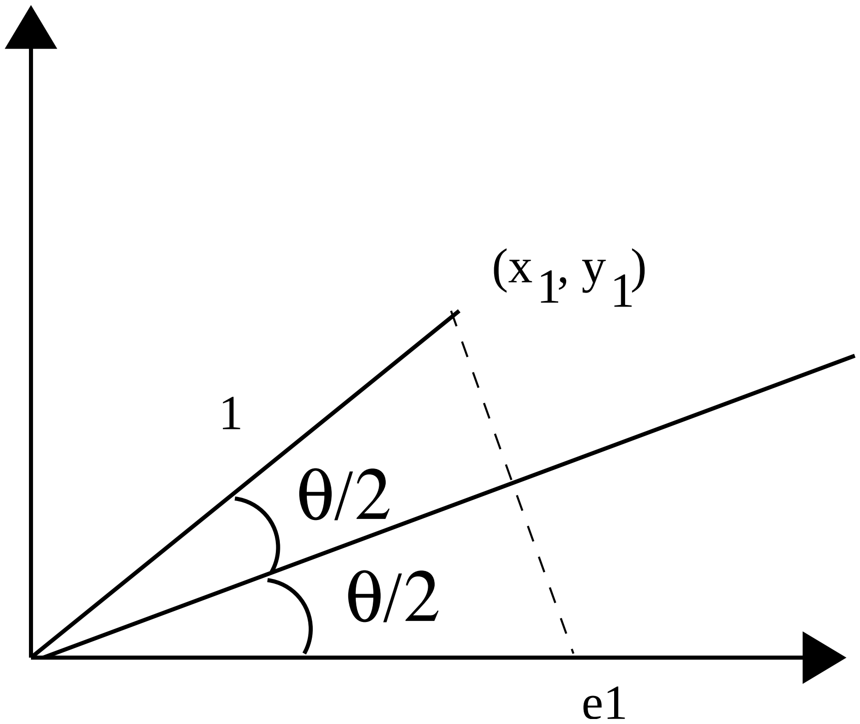


- The circular arc has radius 1, since it passes through e_1 and e_2, which are mapped to (x_1, y_1) and (x_2, y_2) respectively.
- A little trigonometry enables us to calculate the coordinates. $x_1 = \cos(\theta)$ and $y_1 = \sin(\theta)$. $x_2 = -\sin(\theta)$ and $y_2 = \cos(\theta)$.
- Using (1), the matrix representing the rotation is

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

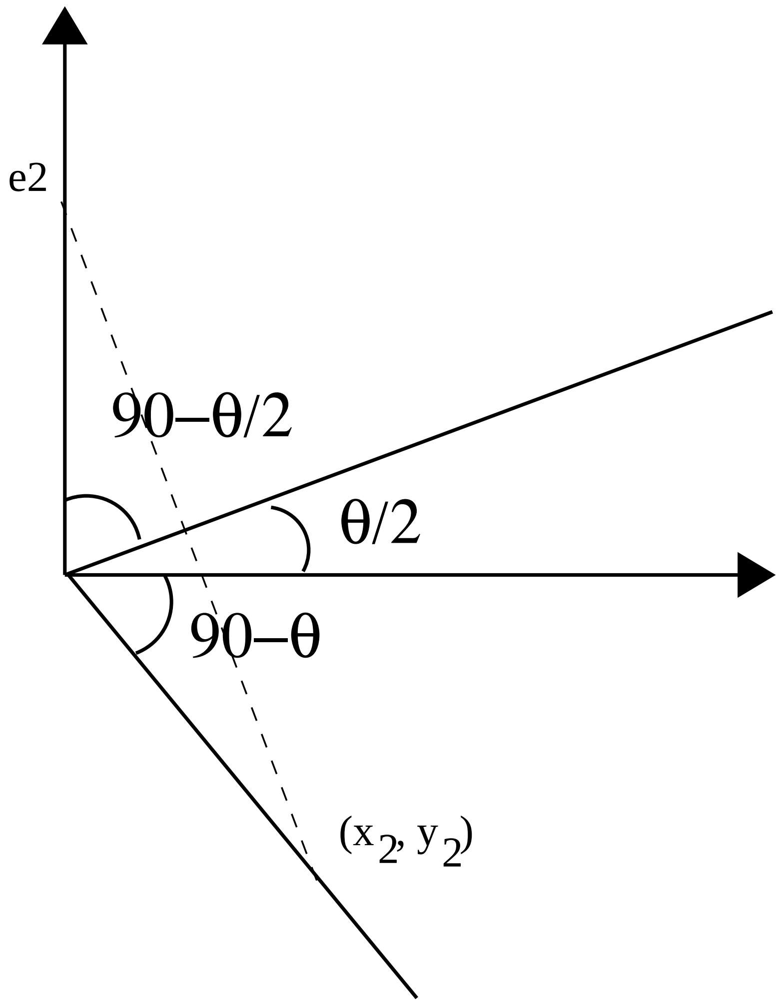
Reflection I

We consider a reflection in a line through the origin at an angle of $\theta/2$.



- The dotted line joins a point with its reflection. It is easy to see that the angles the point and its image make with the reflection line must be the same.
- A little bit of trigonometry shows that the image of e_1 is the same under this reflection as under a rotation by θ, so $(x_1, y_1) = (\cos\theta, \sin\theta)$.

Reflection II



- Some trigonometry shows that $x_2 = \cos(90 - \theta) = \sin\theta$ and $y_2 = -\sin(90 - \theta) = -\cos\theta$.
- Hence the reflection matrix in a line at $\theta/2$ is

$$\mathbf{F} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Exercise

Write down matrices for the following transformations:

1. Reflection in the x-axis.

2. Reflection in the y-axis.

3. Reflection in the line $y = x$.

(Hint: either use the formula or work out the images of e_1 and e_2 directly).

Solution

Write down matrices for the following transformations:

1. Reflection in the x-axis:

$$\theta = 0 \qquad \mathbf{F}_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

2. Reflection in the y-axis:

$$\theta = \pi \qquad \mathbf{F}_\pi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. Reflection in the line $y = x$.

$$\theta = \frac{\pi}{2} \qquad \mathbf{F}_{\pi/2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Homogeneous Coordinates

- All the transformation we are interested in can be represented as matrix multiplication except for one: translation.
- To remove this flaw, we use homogeneous coordinates and treat all transformations in the same way. This representation is used very frequently in computer graphics.
- A point (x, y) in Cartesian coordinates is represented in homogeneous coordinates by a triple (x, y, w), where $w \neq 0$. The corresponding normalised homogeneous coordinates are given by $(x/w, y/w, 1)$.
- In homogeneous coordinates, it is only the ratios between the coordinates that matter, so $(ax, ay, aw) = (x, y, w)$ for any non-zero $a \in \mathbb{R}$. Points with $w = 0$ are called points at infinity, and we won't use them.
- This little piece of mathematical trickery is equivalent to representing the 2D coordinate system in 3D as the plane $w = 1$.

Transformations in Homogeneous Coordinates

In homogeneous coordinates, translation can be written as multiplication by the following matrix

$$\mathbf{p}' = \begin{bmatrix} p_1' \\ p_2' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & v_1 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix} = \mathbf{T_v p}.$$

For any other transformation $\mathbf{M}$, we use the matrix

$$\mathbf{p}' = \begin{bmatrix} p_1' \\ p_2' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix} = \mathbf{Mp}.$$

Composition of Transformations

- Another advantage of using homogeneous coordinates is that it is much easier to work out the effect of composing transformations. This is because they are all represented as matrix multiplications.
- If I perform transformations T_1, T_2, $\ldots T_n$ (in that order) and they are represented by matrices $\mathbf{M}_1$, $\mathbf{M}_2$, $\ldots \mathbf{M}_n$, then the effect of the composition of transformations is to multiply each point by $\mathbf{M}_n \ldots \mathbf{M}_2 \mathbf{M}_1$. (Note the reversal of order).

Example

As an example, suppose that we wanted to rotate an object about some point $\mathbf{u}$. This can be achieved using the building blocks that we know about by:

1. translating object by $-\mathbf{u}$;
2. rotating object by angle θ;
3. translating object by $\mathbf{u}$.

This can be written as

$$\mathbf{T}(\mathbf{u})\mathbf{R}(\theta)\mathbf{T}(-\mathbf{u}) = \begin{bmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -u_1 \\ 0 & 1 & -u_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & u_1(1-\cos\theta) + u_2\sin\theta \\ \sin\theta & \cos\theta & u_2(1-\cos\theta) - u_1\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise

Work out the matrix to rotate by $-\pi/2$ around the point $(-1, -1)$.

Solution

Work out the matrix to rotate by $-\pi/2$ around the point $(-1, -1)$.

1. $\mathbf{u} = (-1, -1)$.
2. $R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
3. The complete transformation is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Summary: Matrices and Linear Transformations

1. A linear transformation in $\mathbb{R}^n$ is defined uniquely by its action on n linearly independent vectors: we can take the unit basis vectors which are zero except for a 1 in the ith coordinate. The image of e_1 is the first column of the matrix, and the image of e_2 is the second column.
2. Translation is defined by adding a vector.
3. Scaling, rotation and reflection can all be defined as matrix multiplication.
4. Homogeneous coordinates represent the 2D coordinate system in 3D as the plane $w = 1$. All the transformations can be written as matrix multiplication in homogeneous coordinates.