Session Objectives

- Write simultaneous linear equations in matrix form
- Solve simultaneous linear equations using Gaussian elimination
- Explain how matrices represent linear transformations
- Write down matrices for some common transformations

Simultaneous Linear Equations

- One important use of matrices is to represent and solve simultaneous linear equations.
- A set of simultaneous linear equations is a group of equations in several variables each of which is linear and all of which must be satisfied at the same time.
- Finding the intersection of two lines, planes, etc. requires the solution of simultaneous linear equations.
- We shall see how an algorithm using row operations can be used to solve such sets of equations.

Matrix Form

When finding the intersection of two straight lines in parametric form we derived the following two equations in two variables s and t:

$$2 - 2t = 5 - 4s$$

$$4 + 2t = 4 - 2s$$

Such equations are usually written with all the variables on the left-hand side and the constants (i.e. the numbers) on the right-hand side:

$$4s - 2t = 3$$

$$2s + 2t = 0$$

This can be rewritten as a matrix product:

$$\begin{bmatrix} 4 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Exercise

Write the following set of equations in matrix form:

$$I_1 - I_2 - I_3 = 0$$

$$3I_2 - 2I_3 = 0$$

$$7I_1 + 2I_3 = 7$$

Solution

Write the following set of equations in matrix form:

$$I_1 - I_2 - I_3 = 0$$

$$3I_2 - 2I_3 = 0$$

$$7I_1 + 2I_3 = 7$$

Solution:

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & -2 \\ 7 & 0 & 2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

Gaussian Elimination

- We solved the equations in the example by a trick: we noticed that adding them together left a single equation in one variable t which could then be solved easily.
- By their very nature, tricks don't work all the time, and what we really want is a general algorithm that can be used to solve arbitrary sets of equations.
- We shall use Gaussian elimination (named after the great 19th century German mathematician Carl Friedrich Gauss) and carry it out using row operations in a systematic way.
- Suppose that the equations are written in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix of coefficients, $\mathbf{x}$ is an $n \times 1$ column vector of variables and $\mathbf{b}$ is an $m \times 1$ column vector of constants.

Row Operations

- Gaussian elimination uses certain row operations to reduce the matrix $[\mathbf{Ab}]$ to echelon form where all the entries below the main diagonal are zero. The two sorts of row operation are:
  1. The solutions are unchanged if we replace any row in A and the corresponding row of the r.h.s. by a linear combination of itself (not multiplied by zero, of course) and any other row.
  2. Interchanging any two rows of A and the corresponding rows of the r.h.s. just rewrites the equations in a different order.
- The basic form of the algorithm uses only the first operation. We divide the first row by a_{11} and zero out the first column. Then divide the second row by a_{22} and zero out the second column etc. The element that we divide by is called the pivot element or simply the pivot.

Back-Substitution

For a set of four simultaneous equations with four variables we reduce A to the following form:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & \alpha_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

Now the equations can be solved by back-substitution:

$$x_4 = \frac{\beta_4}{\alpha_{44}} \qquad x_3 = \frac{\beta_3 - \alpha_{34} x_4}{\alpha_{33}} \qquad \text{etc.}$$

Example

Let us solve the equations

$$\begin{bmatrix} 2 & 1 & -2 \\ 2 & -3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ -3.5 \end{bmatrix}$$

We work with the augmented matrix

$$\begin{bmatrix} 2 & 1 & -2 & -1 \\ 2 & -3 & 2 & 9 \\ -1 & 1 & -1 & 3.5 \end{bmatrix}$$

Stage 1

We use the first row to eliminate the elements in the first column below a_{11}.

1. We divide the first row by a_{11}. Note that if a_{11} were zero, we would swap row 1 with a row without a zero entry in the first column. This gives

$$\begin{bmatrix} 1 & 0.5 & -1 & -0.5 \\ 2 & -3 & 2 & 9 \\ -1 & 1 & -1 & -3.5 \end{bmatrix}$$

2. Take Row $2 - 2 \times$ (Row 1) and Row $3 +$ Row 1.

$$\begin{bmatrix} 1 & 0.5 & -1 & -0.5 \\ 0 & -4 & 0 & 10 \\ 0 & 1.5 & -2 & -4 \end{bmatrix}$$

Stage 2

In stage 2 we work on the second column. We can't use the first row to zero entries in the second column because that would mess up the first column. Instead we use the second row.

1. We divide the second row by a_{22}. Again, if this entry were zero, we would swap row 2 with a later row which had a non-zero entry in the second column.

$$\begin{bmatrix} 1 & 0.5 & -1 & -0.5 \\ 0 & 1 & -1 & -2.5 \\ 0 & 1.5 & -2 & -4 \end{bmatrix}$$

2. Eliminate the entries below a_{22} in the second column.
   Row $3 - 1.5 \times$ (Row 2).

$$\begin{bmatrix} 1 & 0.5 & -1 & -0.5 \\ 0 & 1 & -1 & -2.5 \\ 0 & 0 & -0.5 & -0.25 \end{bmatrix}$$

Stage 3

Divide the third row by a_{32}.

$$\begin{bmatrix} 1 & 0.5 & -1 & -0.5 \\ 0 & 1 & -1 & -2.5 \\ 0 & 0 & 1 & 0.5 \end{bmatrix}$$

The matrix is now in echelon form.

Stage 4

We can now solve the equations easily by back-substitution.

1. From the third row, we see that $z = 0.5$.
2. Now use the second row to find y:

   $$y - z = -2.5 \Longrightarrow y = -2.5 + 0.5 = -2.$$

   Note how we substitute the value of z into the equation to solve it.
3. Now use the first row to find x:

   $$x + 0.5 \times y - z = -0.5 \Longrightarrow x = -0.5 + 1 + 0.5 = 1.$$

We can check the solution by substituting the vector $[1, -2, 0.5]$ back into the original equations.

Exercise

Use row reduction to solve the following system of equations.

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & -2 \\ 7 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

Solution: Echelon Form

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & -2 & 0 \\ 7 & 0 & 2 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 7 & 9 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 7 & 9 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{41}{3} & 8 \end{bmatrix}$$

Solution: Back-Substitution

- From row 3,

$$\frac{41}{3}z = 8 \Rightarrow z = 8 \times \frac{3}{41} = \frac{24}{41}.$$

- From row 2,

$$y - \frac{2}{3}z = 0 \Rightarrow y = \frac{2}{3}z = \frac{2}{3} \times \frac{24}{41} = \frac{16}{41}.$$

- From row 1,

$$x - y - z = 0 \Rightarrow x = y + z = \frac{16}{41} + \frac{24}{41} = \frac{40}{41}.$$

- To check, substitute x, y and z into the original equations. The only one which has changed significantly is the third equation:

$$7x + 2z = 7 \times \frac{40}{41} + 2 \times \frac{24}{41} = \frac{280}{41} + \frac{48}{41} = \frac{328}{41} = 8.$$

Non-unique Solutions

- During the process of Gaussian elimination, we may end up with a row that consists entirely of zeros; if this is the case, then the equations are said to be degenerate.
- If the right-hand side of such a row is non-zero, then the equations are inconsistent and there is no solution.
- If the right-hand side is zero, then the equations are consistent, but there are infinitely many solutions.

Example

Solve the equations

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

The two steps of row operations reduce the augmented matrix to

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in echelon form. The third row is equivalent to the equation $0 = 0$, which is trivially true. If the right-hand side were non-zero, there would clearly be no solution.

- We can still solve these equations by back-substitution, but there will be a variable left undetermined.
- Solving the second equation, we get $y + z = 1$. Substituting for y into the first equation, we get

$$\begin{aligned} x + 2y + z &= 2 \\ \Longrightarrow x + 2(1 - z) + z &= 2 \\ \Longrightarrow x &= z \end{aligned}$$

  Hence the general solution is $[z, 1 - z, z]$ where z can have any real value. This is the parametric equation of a straight line.
- Geometrically, each equation corresponds to a plane (of dimension $n - 1$), and the solution corresponds to the point at the intersection of all the planes. When the equations are degenerate, one plane lies in the intersection of the others, and hence we are left with a line (or a higher dimension linear structure).